Mathematical Entertainments

A Collection of **ILLUMINATING PUZZLES, NEW and OLD**

by **M. H. GREENBLATT**

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To my wife, Nora

Introduction

My reasons for writing this book may very well contain a clue for its understanding and enjoyment.

First, there are a great many beautiful and elegant facts and proofs in mathematics, which are published in widely varying places, if at all. I would like to shout about these problems from the housetops, but have chosen the more dignified, and probably more efficacious, method of writing a book about them.

Second, many people consider mathematics to be a very boring, complicated subject. I hope to show that some facets of mathematics can be quite simple and interesting, although they are not obviously so at first glance.

Third, some of the most beautiful solutions of problems, of which I, alas, am not the originator, do not appear anywhere to the best of my knowledge. Many I am familiar with due only to the tenacious memory and varied experience of M. W. Green.*

All of the problems and notes in this book have, at one time or another, interested me very much, and I would like to think the reader also may be fascinated by them.

Some people may find that many problems appear to be missing. Undoubtedly the collection is incomplete, but all the problems I know of, and consider clever in some

* Research Engineer, formerly at R.C.A. Labs., Princeton, N. J.

sense or another, are represented. Most of them can be done mentally; none of them requires more than a single sheet of paper. Some of the items treated here are well known, but presented in a different way to illuminate a different facet. I earnestly hope this book will be received as it was intended, namely, to furnish pleasant relaxation and to show that mathematics can be beautiful.

No preface is complete without thanks being expressed to the "person without whom this book could not have been written." In the present case, I wish to thank my wife.

M. H. GREENBLATT

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Mathematical Entertainments

1: "Census-Taker" Problems

One of the few amusing things to come out of World War II was a new type of brain twister—the "census-taker" problem. (The time and place of origin of a problem are difficult to specify. To the best of the author's knowledge, this problem was born on the M.I.T. campus in one of the war projects.) Several different versions have appeared, three of which I have labeled the "Neighborhood Census," "The Priest and the Banker," and "The Three Martians."

The Neighborhood Census

A census taker, who was very intelligent, but eccentric in that he *never* asked more questions than necessary, came to a house where three of the inhabitants were not at home. To get their ages, he asked the housekeeper, "What is the product of their ages?"

The housekeeper answered, "1,296."

"What is the sum of their ages?"

"It's the same as the house number." (At this point, we must note that the census taker knew the house number, but you, the problem solver, do not.)

The census taker then said, "I still can't tell their ages. Are any of them older than you?"

When the housekeeper answered, "No," the census taker said, "Now I know all their ages!"

What are those ages?

At first glance this problem appears so "patently incomplete" that a solution seems impossible without invoking necromancy or witchcraft. (My first acquaintance with this problem came in a letter from a friend. I was convinced that his description was incomplete, and I answered by tactfully suggesting a place where the little men in white coats are kind and gentle.) And yet, a logically consistent derivation of a unique set of ages does exist.

In this problem, the product of the three numbers and the sum of the same three numbers were given to the census taker. This information is often sufficient to

n_1	n_2	n_3	*Sum*	n_1	n_2	n_3	*Sum*
1	1	1,296	1,298	2	24	27	53
1	2	648	651	3	3	144	150
1	3	432	436	3	4	108	115
1	4	324	329	3	6	72	81
1	6	216	223	3	8	54	65
1	8	162	171	3	9	48	60
1	9	144	154	3	12	36	51
1	12	108	121	3	16	27	46
1	16	81	98	3	18	24	45
1	18	72	91	4	4	81	89
1	24	54	79	4	6	54	64
1	27	48	76	4	9	36	49
1	36	36	73	4	12	27	43
2	2	324	328	4	18	18	40
2	3	216	221	6	6	36	48
2	4	162	168	6	8	27	41
2	6	108	116	6	9	24	39
2	8	81	91	6	12	18	36
2	9	72	83	8	9	18	35
2	12	54	68	9	9	16	34
2	18	36	56	9	12	12	33

Table 1

determine the three numbers. In this case, the census taker confessed that he still was unable to tell the three ages. Therefore, of the groups of three numbers having the product 1,296 there must be at least two groups with the same sum. *All* groups of three factors of 1,296 are shown in table 1, along with the sums of the groups.

Except in one case, the sum of each group of three numbers is different from the sum of all other groups. This exception is 91, the sum of 1, 18, and 72, and also of 2, 8, and 81. Had the house number been any allowable sum other than 91, the census taker would have been able to choose the unique triad for the ages of the three missing people. But he couldn't. So the house number must have been 91.

Since the last question enabled him to decide which of the two triads was the correct one, the housekeeper must have been between 72 and 81. (Another fact known to the census taker but not to you. For only then would the answer "No" enable the census taker to announce that he knew all the ages—namely 1, 18, and 72.)

The Priest and the Banker

The town priest saw his friend the village banker walking down the street with three girls. Not known for excess verbiage, the priest asked, "What is the product of the ages of these girls?"

The banker answered, "2,450."

"What is the sum of their ages?"

"The same as yours, Father."

"I still can't tell their ages. Are any of them as old as you?"

When the banker said, "No," the priest announced that he knew all the girls' ages.

What are those ages, how old is the priest, and how old is the banker?

The solution to this problem is very similar to that for the "Neighborhood Census," but there are a few new wrinkles. The product of the girls' ages is 2,450. The factors of this number are 1, 2, 5, 5, 7, and 7. Again it is helpful to construct a table showing all the triads that have the product 2,450. Table 2 shows all such triads, along with their sums:

n_1	n_2	n_3	*Sum*	n_1	n_2	n_3	*Sum*
1	1	2,450	2,452	2	7	175	184
1	2	1,225	1,228	2	25	49	76
1	5	490	496	2	35	35	72
1	7	350	358	5	5	98	108
1	10	245	256	5	7	70	82
1	14	175	190	5	10	49	64
1	25	98	124	7	7	50	64
1	35	70	106	7	10	35	52
1	49	50	100	7	14	25	46
2	5	245	252				

Table 2

We can see that only 5, 10, 49 and 7, 7, 50 have the same sum, 64. The priest's age must be 64. Only if the banker was 50 would the priest have been able to choose the correct triad. Therefore, the banker is 50, and the girls are 5, 10, and 49.

The Three Martians

The last twist of this popular type of problem is, in a sense, the trickiest, but still logically consistent. It is said to have been brought back from Mars by an Air Force colonel who visited there. According to him, the Martians are a very intelligent people, partly because they have two heads, as shown in this reproduction of a sketch he brought back.

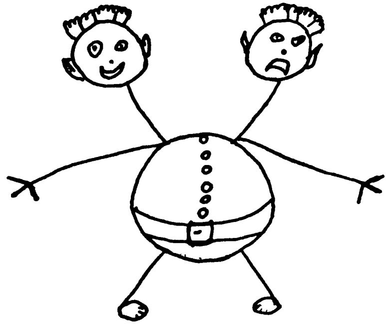


Figure 1

Wanting to know the ages of three Martians, the colonel asked the Chief of Martian Intelligence, "What is the product of their ages?"

"1,252" was the answer.

"What is the sum of their ages?"

"The same as my father's."

"I still can't tell their ages. Are any of them as old as you?"

When the chief said, "No," the colonel announced that he could tell the ages. What are they?

This problem seems so similar to the previous ones that the natural urge is to plunge right in and prepare a table of triads. The factors of 1,252 are 1, 2, 2, and 313.

n_1	n_2	n_3	*Sum*
1	1	1,252	1,254
1	2	626	629
1	4	313	318
2	2	313	317

Table 3

When we inspect this table, we are tempted to exclaim, "Mon Dieu, has Greenblatt flipped or lost his marbles? No two triads have the same sum!"

But, after our passions have subsided, we decide to have another look at the problem. We know we are using the right method to find the answer. We think—perhaps there is something wrong with the numbers! We note from the sketch that the Martian has three fingers on each hand. And, with logic worthy of the great Sherlock Holmes himself, we deduce that the Martians must use the number base 6 because they have six fingers, just as we use the base 10 because we have 10 fingers (5 fingers on each hand). The number 1,252 in the base 6 is equal to:

$$1(6^3) + 2(6^2) + 5(6^1) + 2(6^0)$$
$$= 320 \text{ in the base 10 (also written } 320_{10})$$

It is immaterial whether we form the table of triads in the base 6 or the base 10, because triads having the same sum in one base will also have the same sum in the other base. We construct table 4 in the base 10 because most of us are more familiar with it. The factors of 320_{10} are 1, 2, 2, 2, 2, 2, 2, and 5.

n_1	n_2	n_3	*Sum*	n_1	n_2	n_3	*Sum*
1	1	320	322	2	4	40	46
1	2	160	163	2	5	32	39
1	4	80	85	2	8	20	30
1	5	64	70	2	10	16	28
1	8	40	49	4	4	20	28
1	10	32	43	4	5	16	25
1	16	20	37	4	8	10	22
2	2	80	84	5	8	8	21

Table 4

The fact that the colonel had to ask one more question indicates that the three ages must have the same sum. The triads 2, 10, 16, and 4, 4, 20 both have the same sum, 28. Since the colonel was able to choose between the triads because the chief said none of them was older than he,

the triad in question must have been 2, 10, 16. In the number base 6, the decimal triad 2, 10, and 16 would be 2_6, 14_6, and 24_6, and the decimal triad 4, 4, and 20 would be 4_6, 4_6, and 32_6. The chief's age had to be between 16 and 20 Martian years (or between 24_6 and 32_6 written in his own number system). A Martian year, incidentally, is nearly twice as long as an Earth year, and this accounts for the fact that the chief was so young—less than 32 (Martian years).

2: Diophantine-Type Problems

Diophantine equations are those equations in which only solutions in integers are permissible. For example, the equation $x^2 + y^2 = z^3$ is a Diophantine equation when each of the variables is an integer. One solution is:

$$x = 2, \quad y = 11, \quad \text{and} \quad z = 5$$

which yields:

$$2^2 + 11^2 = 5^3$$

There is another solution to this equation in which x and y are the same, namely:

$$x = 2, \quad y = 2, \quad z = 2$$

which leads to:

$$2^2 + 2^2 = 2^3$$

These are the only solutions (excluding the trivial $x = 0$, $y = 0$, $z = 0$).

Diophantine equations are vastly different from equations which are not restricted to integral values of x, y, and z. The original equation, $x^2 + y^2 = z^3$, would have an infinite number of solutions in continuous variables, but only two with non-zero integral values.

Many of the problems in the following chapters obviously require integral solutions. In this chapter, we will consider other problems which are soluble only by integers.

The Missionary and the 159-Link Chain

One of the simplest problems, which requires no mathematical analysis at all, is about a missionary who had a gold chain with 159 links. He was captured by a group of cannibals (as are most missionaries in mathematical problems), who stipulated that he could purchase one day of freedom for each single additional link of the gold chain that he gave them. They agreed to accept the principle of "change-making" (*i.e.*, he might give them a 7-link section, if they had 6 links to give him in change). The missionary's goldsmith charged an exorbitant rate for making each cut link whole again. What is the minimum number of links that he had to cut in order to assure his existence for 159 days (at which time he knew that he would be rescued)?

The solution to this problem is simplified if we recognize that by cutting a single link, he could obtain two smaller chains and a single cut link. By making only four cuts, he produced small chains of length 5, 10, 20, 40, and 80, and four single links. These enabled him to live until the plane came.

Restored Arithmetic Problems

The previous problem hardly requires Diophantine analysis. A better example of an elementary Diophantine problem might be one in the class known as "restored arithmetic." In these problems some or all of the digits have been erased and are represented by dots or letters. The problem solver is asked to find digits consistent with the rest of the problem. The answer requires the solution of many small Diophantine equations. Some information is always given in these problems, and it is natural to wonder how "degraded" this information can be. Problems in which only one digit is given are known. W. W. R. Ball

gives an example* of a restored division problem with no digits given. The only information is that the quotient of the problem is a repeating decimal.

We can make up a simple problem: $\overset{x}{\sqrt{xx}}$. Please note that this seems to contain no information at all, until we state that it is a fifth root extraction. The only two-digit integer whose fifth root is a single integer is 32.

Figure 2 shows an original problem in restored square root extraction. The only information given is that the answer is an even number. The answer is 8,604.

$$
\begin{array}{rrrr}
x & x & x & x \\
\hline
\sqrt{xx} & xx & xx & xx \\
xx & & & \\
\cline{1-1}
xx & xx & & \\
x & xx & & \\
\cline{1-2}
 & x & xx & xx \\
 & x & xx & xx \\
\cline{2-4}
 & \cdot & \cdot\cdot & \cdot\cdot
\end{array}
$$

Figure 2

Rectangular Block Cut into Unit Cubes

The next problem is: A rectangular parallelepiped, of dimensions *abc*, where *a*, *b*, and *c* are integers, is painted red on the outside. The block is then sawed into unit cubes, and it is found that the number of cubes having no paint at all is exactly equal to the number of cubes having some paint on them. What are the allowable values of *a*, *b*, and *c*?

The total number of unit cubes is *abc*, and the number of cubes in the "inner core" (which have no paint on them) is the volume of the rectangular parallelepiped one unit in

* Ball, W. W. R., *Mathematical Recreations and Essays*, New York: The Macmillan Co., 1962, 11th edition, pp. 22 ff.

from each edge, or two units less in each dimension. We therefore want solutions for which

$$abc = 2[(a - 2)(b - 2)(c - 2)]$$

An $8 \times 10 \times 12$ original block will have an inner core of size $6 \times 8 \times 10$. The ratio of these two volumes is accordingly $8 \times 10 \times 12$ divided by $6 \times 8 \times 10$. The eights and the tens cancel each other, and we are left with a simple ratio: $\frac{12}{6}$ which, indeed, equals 2.

Further solutions cannot be seen in this simple manner. A straightforward way to derive these other solutions is to set up the equation:

$$\frac{abc}{(a - 2)(b - 2)(c - 2)} = 2$$

Dividing by abc, and inverting, we obtain:

$$\left(1 - \frac{2}{a}\right)\left(1 - \frac{2}{b}\right)\left(1 - \frac{2}{c}\right) = \frac{1}{2}$$

Now we can try various integers for a, b, and c. Suppose, for example, we arbitrarily choose $a = 5$. Then the first fraction in parentheses is equal to $\frac{3}{5}$. To get rid of the 5 (so that we may ultimately obtain the fraction $\frac{1}{2}$) we must choose for b a number which is 2 greater than a multiple of 5. Twelve is one such number, but it is not large enough, because the second parenthesis would be $\frac{10}{12}$ ($\frac{5}{6}$). The first two fractions are now equal to $\frac{3}{5}$ and $\frac{5}{6}$, and the product is already equal to $\frac{1}{2}$, so that pair is not allowable. Let us try $b = 17$. The fraction in the second parenthesis is now $\frac{15}{17}$, and the total fraction so far is $\frac{9}{17}$, still more than $\frac{1}{2}$. We must now try to multiply this fraction by a suitable $(c - 2)/c$ so that the product will equal $\frac{1}{2}$. Again, we see that $c = 36$ will have a very happy result because $(c - 2)/c = \frac{34}{36}$ and this fraction, multiplied by $\frac{9}{17}$, equals $\frac{2}{4}$ or $\frac{1}{2}$. This method of solution is the best that I can offer. These Diophantine problems do not have the

same quality of generality that I try to stress in the succeeding chapters. I feel almost apologetic about including them, but they are interesting. Perhaps some reader will be able to furnish the generality that my exposition lacks.

Some additional values for a, b, and c are given in table 5. There are about ten more solutions, and the reader

a	b	c
5	13	132
5	18	32
$\underline{6}$	$\underline{10}$	$\underline{32}$
6	11	24
$\underline{6}$	$\underline{12}$	$\underline{20}$
$\underline{6}$	$\underline{14}$	$\underline{16}$
7	7	100
$\underline{8}$	$\underline{8}$	$\underline{18}$
8	9	14
$\underline{8}$	$\underline{10}$	$\underline{12}$

Table 5

may amuse himself by finding them if he wishes to. The underlined values are thought to be the only sets consisting solely of even numbers.

Basketball Scores

A basketball team thought one point for a free throw and two points for a basket weren't really commensurate with the difficulty in achieving them. So they decided to award a points for a free throw, and b points for a basket. After this was done, they discovered there were precisely 35 scores that could not be attained, and one of these scores was 58. What were the values of a and b?

We will not even consider the pedestrian method of simply trying different numbers for a and b. After

playing around with this problem and several sets of numbers, we discover that a and b must be relatively prime (have no common factor). We can also discover that $(a - 1)(b - 1)/2$ scores cannot be achieved.

The proof of this statement will not be given but is "left as an exercise for the interested reader." With the information given we can write that $(a - 1)(b - 1)/2 = 35$. Knowing that 58 is one of the unattainable scores, deep concentration and soul searching will show that 8 and 11 are a suitable pair of numbers. Further investigation indicates that these form a unique solution. In this example, as in the last, the lack of generality in the solution of Diophantine equations is quite apparent.

The Incorrectly Cashed Check

Our next problem is about the man who received a paycheck for so many dollars and so many cents. He cashed the check at a bank where the teller made the unlikely mistake of giving him as many dollars as there should have been cents and as many cents as there should have been dollars. Our hero did not notice the error, but went forth and made a five cent telephone call (you can see how old this problem is). He then discovered that he had precisely twice as much money left as the check had been written for. What was the amount of his paycheck?

This problem allows a somewhat more rigorous solution than the previous ones. We recognize, first of all, that the number of dollars on the original check must have been less than 100 for the teller to have made the mistake which we say he did. If we assume that the original check was for r dollars and s cents, the incorrect sum must have been $(100s + r)$ cents. Now we can write the equation:

$$
\begin{aligned}
100s + r - 5 &= 2(100r + s) \\
100s + r - 5 &= 200r + 2s \\
199r - 98s + 5 &= 0
\end{aligned}
$$

s must be approximately twice as large as r, so let us write $s = 2r + x$ where x is some, hopefully small, correction. Substituting this new value for s, we have:

$$\begin{aligned} 199r - 98(2r + x) + 5 &= 0 \\ 199r - 196r - 98x + 5 &= 0 \\ 3r - 98x + 5 &= 0 \\ 3r &= 98x - 5 \end{aligned}$$

We know that r is a whole number less than 50. Therefore, the expression $98x - 5$ (which is equal to $3r$) must be less than 150 and must be divisible evenly by 3.

If x were equal to 2, $98x - 5$ would be greater than 150. If x were equal to 0, $98x - 5$ would be negative—an impossible solution. Therefore, x must be equal to 1. Continuing:

$$\begin{aligned} 3r &= (98 \times 1) - 5, \text{ or } 93 \\ r &= 31, s \text{ then equals } 63 \end{aligned}$$

The original check must have been for $31.63

On Moving Day

A friend of mine, while we were graduate students, was quite a decent student, but was extremely absent-minded. However, he wasn't quite as bad as the hero of the following anecdote. A late, very well known physicist was exceptionally brilliant in technical matters, but was helpless when it came to matters concerning the running of a house. So bad was this situation that his wife had to run all household affairs. One day, they were going to move into a new house. When he left for work that morning, his wife asked him to please remember to come home to their new house, since no one would be home at the old house. In order to have a story, he returned to the old house, of course. He strode up to the front door, firmly expecting to find it unlocked. When it refused to yield, he was very puzzled. He went around to the side and

peered in a window. Seeing the house completely empty, he said, "What a fool I've been! My wife told me this morning that we were moving to the new house, and I should have remembered. But now—I wonder where we moved to?"

Just then he saw a small girl walking down the street. With a flash of inspiration, he stepped to the sidewalk and said to her, "Pardon me, little girl. The Browns used to live in this house, but I understand they moved today. Can you tell me where they moved to?"

"Yes, Daddy," she replied. "Mommy sent me here to get you!"

Expression for Which 2,592 Is a Solution

This friend of mine from graduate school was nearly as absent-minded as Dr. Brown. One day he came into our room while Larry and I were minding our own business (studying, of course) and asked, "Say, fellows, how am I going to remember my home phone number? It's EVergreen 2592.* I can remember the exchange all right, but the numbers confuse me. Isn't there some easy way for me to remember them?"

I considered the question an idle one and promptly forgot about it. Several hours later, Larry, drawn, haggard, and with bloodshot eyes, started muttering to himself, "No, it can't be, it isn't, but it *is*!" Then he gleefully explained, "The number 2,592 appears to be a unique solution in integers of the expression $x,yzx = x^y z^x$, where the x, y, and z represent the same digits in the same order on both sides of the equation (a decimal expression on the left and the powers of integers on the right), and are different from each other."

* This is a true story. The phone number in question was the phone number of B. Rothlein, who lived on 41st Street in Philadelphia between 1943 and 1947.

Larry's method for discovering that 2,592 is a unique solution of the above expression is completely unknown to me. I'm not even sure of a concise way to prove its uniqueness.

Railroad Trestle Problem

Two boys were crossing a railroad trestle when they suddenly heard the sound of an approaching train. Being prudent, they started to run for safety toward opposite ends of the bridge. Running at the same speed, each reached his particular end of the bridge just in time to avoid being hit by the locomotive. The boys were $\frac{2}{5}$ of the way across when they heard the sound of the whistle. The train was going 50 mph. How fast did the boys run?

In solving this problem, we must recognize that the time taken by the train to cross the bridge was equal to the difference in times at which the boys arrived at their respective ends. If the boys were $\frac{2}{5}$ of the way across when they heard the whistle, then one had a distance $\frac{2}{5}L$ to run (where L is the length of the bridge) and the other had to run $\frac{3}{5}L$. The difference in the distances that the boys ran is $\frac{1}{5}L$, and if they ran at a velocity V_B, then the time difference was $\frac{1}{5}L/V_B = L/5V_B$. This must be equal to the time that the train took to cross the bridge, or L/V_T. Equating these two expressions:

$$\frac{L}{5V_B} = \frac{L}{V_T} \quad \text{or} \quad V_B = \frac{V_T}{5}$$

The boys therefore ran at a velocity of 10 mph.

Integral Pythagorean Triplets

The discovery of integral Pythagorean triplets can be fascinating. (Pythagorean triplets are integers a, b, and c, which are related by the expression $a^2 + b^2 = c^2$. If the

value of c is given, there are infinitely many solutions for a and b in nonintegers, but there may be only a few in integers.)

The squares of two consecutive integers, n and $n + 1$, differ by $2n + 1$. If $2n + 1$ is, itself, a square, we will have the Pythagorean triplet $\sqrt{2n + 1}$, n, $n + 1$. For example: if $2n + 1 = 25$, then we would have the Pythagorean triplet 5, 12, 13 because the difference between 12^2 and 13^2 is 25. An infinite set of Pythagorean triplets can be built up in this way.

An interesting problem can be mentioned here. I once casually asked my children how many integral Pythagorean triangles they could name, in which one edge was 15 units long. The answer is that the 3, 4, 5 right triangle can be multiplied by 3 to give a 9, 12, 15 right triangle, and by 5 to give a 15, 20, 25 right triangle. Then, since 15^2 is 225, we can write: $225 = 112 + 113$, and so 15, 112, and 113 is such a triangle. By somewhat more exotic methods, we discover that 8, 15, 17 is such a triplet. We can multiply the 5, 12, 13 triangle by 3 to get 15, 36, 39. The answer is that there are five such triplets.

If we choose any two numbers, u and v, then the number $u^2 - v^2$ and the number $2uv$ form the two lower numbers of a Pythagorean triplet. The other number is $u^2 + v^2$. The reason is that $(u^2 - v^2)^2 = u^4 - 2u^2v^2 + v^4$, $(2uv)^2 = 4u^2v^2$, and the sum of these two is:

$$u^4 + 2u^2v^2 + v^4 = (u^2 + v^2)^2$$

All primitive Pythagorean triplets can be formed this way.

A more difficult problem has been suggested in the *American Mathematical Monthly* (1914). It asks to find three integers such that the sum of the squares of any two of them is, itself, a perfect square. The solution to this problem is involved, but not difficult. The smallest numbers are 44, 240, 117, and we refer people interested in a demonstration to the original source.

3: Königsberg Bridges, Design Tracing, and Euler

Story of Seven Bridges

In Germany, there is a town called Königsberg, situated on the banks of the Pregel River. The island of Kniephof is in the river, just below another island, and both islands and the banks are connected by bridges as shown in figure 3:

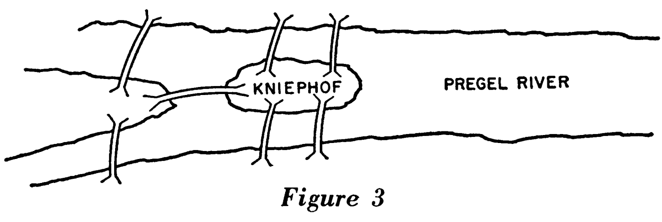


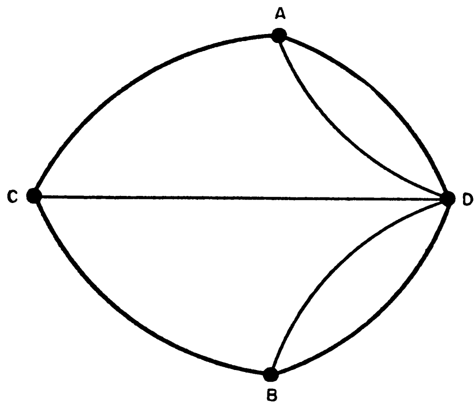
Figure 3

These are the seven Königsberg Bridges. And thereon hangs a tale. The residents of Königsberg used to wonder if it was possible to walk and cross each of these bridges once and only once. Try as they would, they never succeeded, but they didn't know if their failure was due to their limited imaginations or to the nature of the problem.

Word of their dilemma finally reached the ears of the great mathematician Leonhard Euler. He offered a solution to the problem which is beautiful in its simplicity.

First he replaced the islands and land masses by points, and the bridges by lines—as in figure 4.

Figure 4



Following Euler's method we can characterize each junction of lines (or bridges) as odd or even, depending on how many lines emanate from it. If all junctions are even, we can traverse the entire array, going over each line once only, because for every line entering a junction, there will be another line leading out. If there are, at most, two odd junctions, the complete traversal can still be made, provided we start at one of the odd junctions and end at the other. If the array has more than two odd junctions, then no power on earth can accomplish the traversal in one continuous path. In figure 4, all four junctions are odd. (The number of odd junctions, incidentally, must always be even. The proof of this is very similar to the proof that the number of people shaking hands at a conference an odd number of times must be even. This problem is considered later in this chapter.)

If we insert another bridge from A to B, two of the junctions (A and B) will become even, and the other two will remain odd. A complete journey could then be made by starting at C or D. Without the additional path from A to B there are four odd junctions, and therefore all the

paths can not be traversed in a single journey. All paths can, however, be traversed in $n/2$ separate journeys, where n is the number of odd junctions. The Königsberg Bridges, therefore, can not be traversed in a single journey, but they can be traversed in two separate journeys.

Design Tracing

The solution of the Königsberg Bridge problem is directly applicable to a class of problems known as design

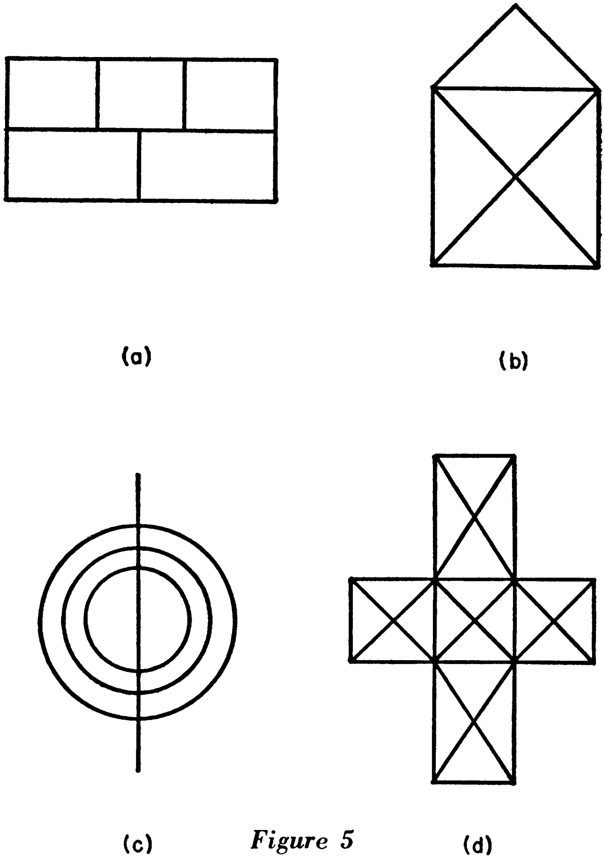


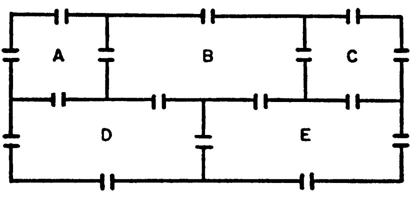
Figure 5

tracing problems. Figure 5 shows some typical examples of drawings for this type of problem.

In each case, the question is: Can this design be traced in one continuous line (*i.e.*, without lifting the pencil from the paper, or retracing any line)? Problems such as these can always be done if there are, at most, two odd vertices. Otherwise, it would take $n/2$ distinct lines or tracings. The drawing in figure 5(*a*) has eight odd vertices, (*b*) has two odd vertices, (*c*) has two odd vertices (the ends of the vertical line), and (*d*) has twelve odd vertices. Part (*a*) requires that we lift the pencil four times. Part (*b*) can be drawn in one continuous line if we start at one of the lower corners and end at the other lower corner. Part (*c*) can also be drawn in one line if we start at the top or bottom. Part (*d*) requires six separate tracings.

Slight variations on this design tracing problem are popular. They are usually subject to the same analysis as we have considered. For example, a problem often given is shown in figure 6:

Figure 6



The outline shows a five-room diagram. There are doors as shown in the drawing. The question is: Can a walk be planned so as to pass through each doorway once and only once? Inspection shows that rooms A and C have an even number of doors leading into them, but rooms B, D, and E have an odd number of doors leading to them. So the required walk cannot be performed. If the top wall in room B had another door, then there would be only two odd rooms, and the walk could be performed, if we start in D or E (and end in E or D).

Weird stories concerning "a person that a friend of mine knows," who has solved a problem such as the one in figure 6, occur very frequently. Invariably, the time of the day, or the season, or the phase of the moon, or unsettled world conditions are such that we cannot see the "solution." The existence of such stories must be accepted as par for the course in this field of impossible constructions.

Whenever an outline can be traced in one continuous line, that line can be drawn so that it never crosses itself. A method for doing this has been proposed by T. H. O'Beirne. A full explanation can be found in the April 1964 issue of *Scientific American.**

Parity of Number of People Shaking Hands at a Conference

Another problem susceptible to the treatment given to the preceding problems is the one about people shaking hands. At a recent conference, the members shook hands with each other an arbitrary number of times. Prove that the number of people who shook hands an odd number of times must be even.

The answer is easy to see if we consider a drawing in which each person is represented by a dot, and each handshake by a line joining two dots. Each dot can be labeled O or E, depending on whether there are an odd or even number of lines coming from it. Each new line that is drawn must necessarily go from an O to an O, or an E to an E, or an O to an E. A line going from O to O changes both of them to an E and thus decreases the number of O's by an even number—2. A line going from E to E changes both to an O, and this, again, changes the number of O's by 2. A line from E to O changes the E to an O, and vice versa—the number of O's is unchanged. Since the number of O's can only change by an even number,

* M. Gardner, *Scientific American,* April 1964, p. 126.

and the process must start with zero O's (an even number) —the number of O's must remain even.

Euler's Formula

Another beautiful derivation by Euler relates the number of edges (E), the number of countries (C), and the number of vertices (V) in any map on a plane surface. (The formula relating E, C, and V on the surface of a sphere is almost identical.) This formula can be presented in a very simple manner—a way which makes hardly any mathematical demands beyond a keen interest. Consider the map in figure 7a:

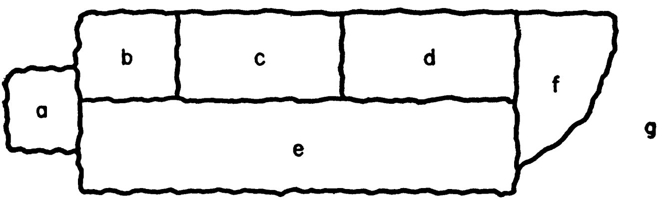


Figure 7a

Let us consider the area outside the group of countries, labeled g, to be an ocean. Each of the boundaries is a dike, which prevents the country from being flooded. Now break down the dikes systematically, so as to flood each country, one at a time. But do not break any dikes except to flood a new country. The unbroken dikes are then all connected to each other. (For if two groups were not so connected, then the last dike joining the two groups should not have been broken, because there was water on both sides, and no new country was flooded thereby.)

Figure 7b shows the same map as figure 7a, but dotted lines indicate the dikes broken to flood each country. In addition, a large dot is placed at each vertex where three or more edges met.

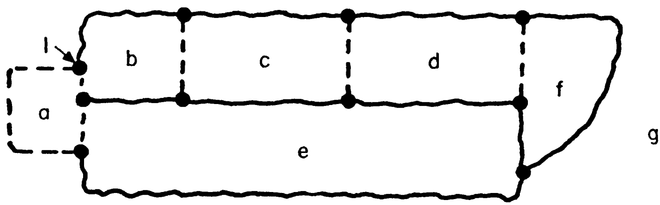


Figure 7b

In our derivation, we will represent the number of broken dikes by B, and the number of unbroken dikes by U. The number of broken dikes must exactly equal the number of countries, C. For at least one dike per country must be broken, and there is no need to break more. The unbroken dikes are all connected, as was mentioned. And if we start at a vertex such as the one labeled 1 in figure 7b, then each unbroken dike will have an additional vertex at its end. Thus, we have $B = C$, and $V = U + 1$. Since the total number of edges (or dikes) must be equal to $B + U$, we can write:

$$E = B + U = C + V - 1$$

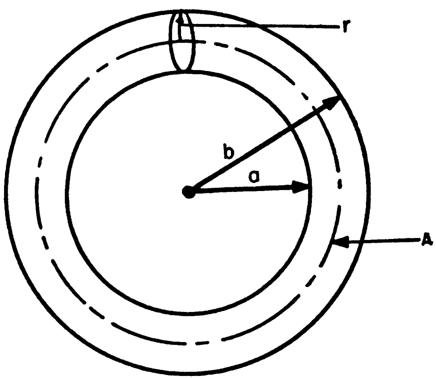
This is Euler's formula. On the surface of a sphere, the same relation holds, except that C increases by one, since g in figure 7a is another country. To account for this, the formula changes to $E = C + V - 2$. This formula forms the basis for a proof that there are no more than five regular polyhedra (the so-called Platonic solids). It is also used in a proof that five colors are sufficient (but not that they are necessary) to color any map, so that each country may be colored differently from countries adjacent to it. It can still not be proved that four colors are sufficient on a plane. (On the surface of a torus, or doughnut, it can be proved that seven colors are both necessary *and* sufficient.)

4: Bits and Pieces

Volume of an Anchor Ring

How can we determine the volume of an anchor ring, as shown in figure 8?

Figure 8



Pappus' Theorem, which is a part of integral calculus, can be used to determine the volume. But the method described here is considerably simpler, and exact. Consider the ring cut in half as shown in figure 9(a), and the lower half rotated from the position indicated by the dotted lines.

Then, rotate the portions outside the vertical lines shown in figure 9(a), so that the ring now resembles

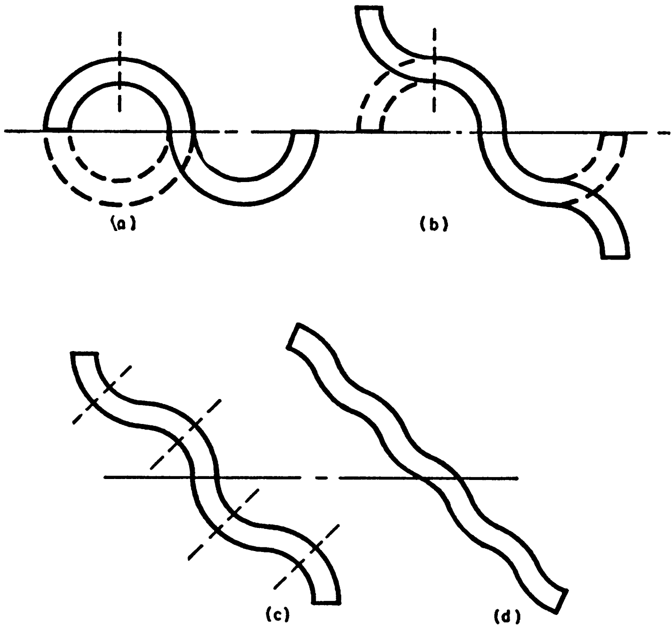


Figure 9

figure 9(b). Again, the portions at each of the lines in figure 9(b) can be rotated to resemble figure 9(c). At each step, the volume of the original anchor ring is not changed at all—it is merely rearranged. The process described will eventually result in a straight circular cylinder whose cross-sectional area is πr^2, and whose length is the circumference of the dotted line at A in figure 8, namely $2\pi(a + b)/2$. The ends of the cylinder are perpendicular to the axis of the cylinder because they were originally cut "square" and were never changed after that. The volume of the resultant tube is therefore:

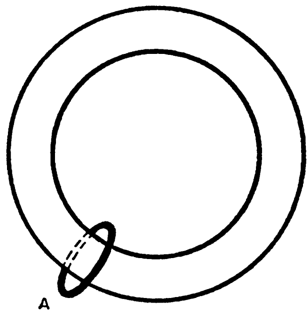
$$\pi r^2 \times \pi(a + b) = \pi^2 r^2 (a + b)$$

Q.E.D.

Topological Behavior of a Loop Around a Torus

The next problem could belong in a chapter in which the obvious solutions are wrong. Consider a rubber inner tube. Can this tube be turned inside out through a hole at the outer edge, and still be, topologically, an inner tube? The simple, but treacherously incorrect, solution considers an inner tube as shown in figure 10. A string is

Figure 10



wrapped around the smaller diameter as is shown at A. It is inextricably looped with the tube, and no simple deformation can change its "loopedness." The only way to unlink the string from the tube would be to cut one of them. But if it were possible to turn the tube inside out and still have a tube, then the string at A would be on the inside, and one would assume that we could reach in through the hole and pull out the string. Since the "loopedness" cannot be so easily changed, it seems impossible to turn the tube inside out and still have a tube. But as stated earlier, this argument is fallacious.

Figure 11 shows the inner tube in six stages during the deformation.

Figure 11(a) shows the inner tube before deformation. In (b), the hole shown at the left of the drawing in (a) is tremendously enlarged. In (c) and (d) the edges of that

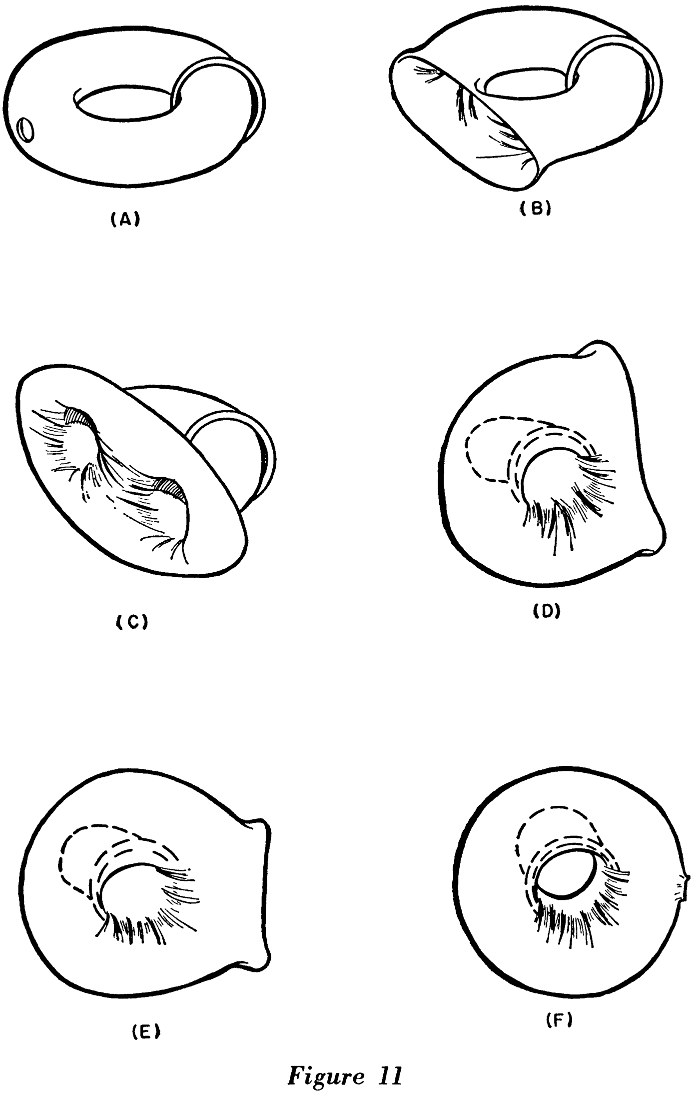


Figure 11

hole are folded back, and if the large circle is now allowed to shrink back to its original size, as in (*e*) and (*f*), the operation is complete. (*f*) resembles (*a*) if (*a*) is rotated 90° about a horizontal axis. But the string which was tied around the "small" diameter is now around the "large" diameter—around the inside of the tube! This is why the original argument was wrong. It *is* possible to turn an inner tube inside out and still, topologically, have an inner tube.

Which Raise Nets More Money?

The next problem can be relied upon to start an argument. A man is offered a job at $1,000 a year, and is offered increases of $50 more each half year or $200 more each year. Which choice will net more money? The choice seems so obvious that there must be a "hooker" somewhere. But, I assure you

(*a*) there is no hooker, and

(*b*) the obvious solution is wrong.

Let us construct a table showing the amounts of money earned in each case:

	$50 raise each half year		*$200 raise each year*
1st half year	500		
2nd half year	550		
Total, 1st year		1,050	1,000
3rd half year	600		
4th half year	650		
Total, 2nd year		1,250	1,200
5th half year	700		
6th half year	750		
Total, 3rd year		1,450	1,400
Total received over 3 years		$3,750	$3,600

Table 6

The raise of \$50 every half year is actually more rewarding than the alternative. (I don't think that this problem should be used as a proof that "all that glitters is not gold.")

Our intuition may have led us astray because we didn't take time to analyze the problem. Everybody can figure out what "\$50 more every half year" means in terms of a year, but few people recognize immediately that \$50 per half year is the same as \$100 for a full year, and further a raise at this rate occurs twice a year! Thus, the two alternatives are raises at the same yearly rate, but the half-yearly raise starts earlier.

Stacking of Discs

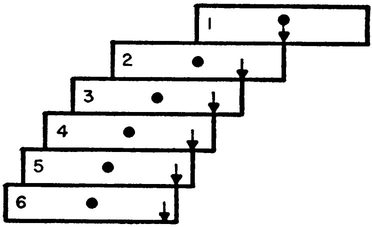
You are given a large number of discs, and asked to place them one on top of the other, displacing each one horizontally by as much or as little as you please. What is the maximum horizontal displacement of the top disc from the bottom disc you can obtain before the stack topples?

This is another problem for which an automatic reaction can mislead. The usual answer is that the top disc can surely not be displaced by more than one radius (or maybe, a diameter) from the center of the bottom disc. As is customary in this book, the instinctive answer to a simple question is wrong.

The simplest way to visualize the correct solution is to consider the arrangement of the discs from the top going down. The drawing in figure 12 shows the arrangement of the top six discs, each being horizontally displaced as much as possible.

The radius of each disc is r, and the center of gravity of the disc is represented by the dot in the center. The edge of disc 2 is placed under the dot of disc 1. The center of gravity of the first two discs is now $r/2$ in from the

Figure 12



edge of disc 2 (as is shown by the arrow on the lower edge of 2) and the edge of disc 3 must be put under that point. The center of gravity of any particular group of discs must, for stability's sake, be placed on the next disc below it. In order to achieve the greatest possible displacement, it is most advantageous to put it directly at the edge. The center of gravity of the three discs is $r/3$ in from the edge of disc 3, and the edge of disc 4 must be placed under that point. The edge of disc 5 is displaced by $r/4$, and that of disc 6 by $r/5$, etc. The displacements form the series r, $r/2$, $r/3$, $r/4$, etc., and the total displacement (where n is the total number of discs) is thus:

$$\text{Displacement} = r(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + 1/n)$$

This is the harmonic series, which is known to diverge. The top disc can thus be displaced from the bottom one by as much as we wish!

It is amusing to assemble ten pennies in this manner. The sum of the first ten terms of the harmonic series is nearly 3, and with only ten pennies, the bottom penny can be completely out from under the top one.

Peculiarity of 142,857

Consider the number 142,857. If it is multiplied by 2 or 3 or 4 or 5 or 6, the result is a cyclic permutation of the original number. For example, $142{,}857 \times 3 = 428{,}571$. The usual question is: Can this be easily explained? It can, and the reason is that 142,857 is the repeating portion

of the decimal expansion of the fraction $\frac{1}{7}$, and the periodicity (6) is one less than the reciprocal of the fraction itself (7). This means that the remainders, on performing the required division, must have six different values. (If the remainders were ever the same as a previous one, then the repeating portion of the decimal would start right there.) But since the repeating portion of the decimal has a periodicity of 6, and each remainder must be less than 7 and cannot be 0, each number from 1 to 6 must occur just once per cycle in the sequence of remainders. When the remainder is 2, then the sequence of numbers in the quotient is exactly the same as it would be if we were expanding $\frac{2}{7}$. And similarly, when the remainder is 3, the sequence must be the same as that for $\frac{3}{7}$. So the sequence for an integral number of sevenths must be the same in each case.

Seven happens to be the only single digit (not counting 2) whose reciprocal is a repeating decimal of periodicity one less than the number. There are other numbers whose reciprocal has a periodicity one less than the number itself, but these numbers are large, and hence, the demonstrated multiplication doesn't appear quite as startling. (17 and 19 are the next two such numbers.)

Hempel's Paradox

Part of the discipline, logic, represents classes of objects by suitable symbols. For example, the class A might represent the class of all four-footed objects: chairs, tables, dogs, etc. The logical statement that all A is B is equivalent to the statement that all not-B is not-A. This is not very surprising if we let A represent the class of all normal dogs, and let B represent the class of four-footed animals. Then all A is B says that all normal dogs are four-footed animals. And the phrase "all not-B is not-A" becomes "All not-four-footed animals are not-normal dogs."

The logical statement seems to assume a ridiculous meaning if we take the sentence "All crows are black" and transform it into "All not-black objects are not-crows." If one were to try to prove this statement experimentally, then he could say, "Here is a red pencil," and that is, indeed, a not-crow. And "Here is a blue book," and it, too, is a not-crow. And "Here is a white cup," and that, also, is a not-crow. It begins to look as if all crows *are* black. Now this line of reasoning appears to be preposterous and useless.

But suppose we wish to investigate the truth of the statement that all red-headed secretaries at a particular company are married, and assume, further, that this investigation is to be done at a place where there are one thousand secretaries, three hundred of whom have red hair, and one hundred of whom are not married. We can assign the class of red-haired secretaries to A and the class of married secretaries to B. To investigate the statement that all red-headed secretaries are married, we would have to ask all three hundred red-headed secretaries if they *were* married—three hundred questions. On the other hand, if we were to investigate the equivalent statement "All not-married secretaries are not red-haired," then we have only one hundred people to question. It is obviously more economical to use the inverted form of the logical statement even though it appeared ridiculous in the black-crow example. This seeming inconsistency is known as Hempel's Paradox.

Geometric Proof That

$$\sum_{1}^{N} m^3 = \left[\sum_{1}^{N} m\right]^2$$

The following proof was brought to my attention by M. W. Green, who attributes it to S. Golomb. Discovery

of this proof requires the ability to visualize the geometrical construction shown in figure 13. In this figure, in the upper left corner, there is a unit square which represents the area 1_A^2. To the right of that square is square 2_A^2, and below it is square 2_B^2. To the right of square 2_A^2 is square 3_A^2. Below square 2_B^2 is square 3_C^2, and along the diagonal is square 3_B^2. These three squares of size 3^2 each, have a combined area of 3×3^2 or 3^3. We can continue drawing the squares as we have done until now, and the drawing will resemble the drawing shown in figure 13.

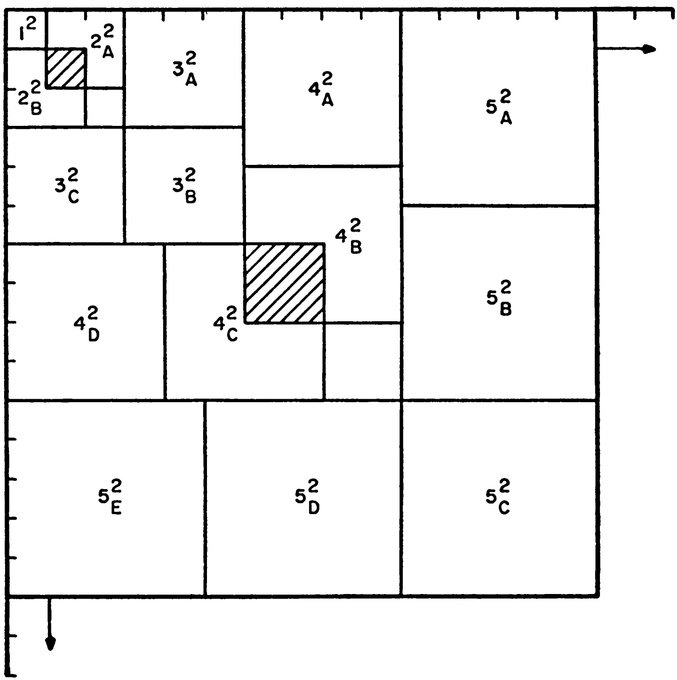


Figure 13

All the squares fit together very nicely except for the even squares along the diagonal. Squares 2_A^2 and 2_B^2 overlap

along the diagonal. But the area of overlap, which is shaded on the drawing, is equal to the vacant area next to it along the diagonal. Because of this beautiful interleaving, we can write that the total area of the squares is:

$$\begin{aligned}\text{Area} &= 1(1^2) + 2(2^2) + 3(3^2) + 4(4^2) \ldots \\ &= 1^3 + 2^3 + 3^3 + 4^3 + \ldots\end{aligned}$$

This area is also equal to $[1 + 2 + 3 + 4 + \ldots]^2$. Since the two areas are equal, this completes the demonstration and none but mathematical purists would deny that it is an effective and beautiful demonstration.

In deference to the purist, it must be recognized that there are dangers inherent in too easy a generalization. Some of the people who refused to accept the "obvious" statement that parallel lines never meet must be given credit for the development of non-Euclidean geometries. But it is difficult to determine *when* the questioning of an obvious statement is profitable and when it is not.

These thoughts bring to mind the story of the experimental physicist who set out to test the conjecture that all odd numbers are prime. He started by saying, "Three—that's prime. Five—that's prime, too. Seven—still prime. Nine—oops!—What's wrong? But let's go on a bit further. Eleven—that's prime. Thirteen—still prime. Nine must have been an experimental error!"

A Peculiar Infinite Sequence

Consider the infinite array

$$S = x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}} \underset{\to}{} \infty$$

For what values of x is S finite? People unfamiliar with this problem are usually tempted to say that S diverges for all values of x greater than one, and approaches zero for all x less than 1. But such is not the case. It can be shown easily, by means of a "log-log" slide rule, or by

other means well-known to the mathematician, that $1.2^{1.2} = 1.244$, and $1.2^{1.244} = 1.254$. (Please note that for any value of x, S must be evaluated from the top down.) In fact, $S = 1.258$ when $x = 1.2$. And we can find finite values of S for $x < 1$. For example, if $x = .5$, $S = .641$. Table 7 gives the value of S for representative values of x in the range $0 < x < 1.445$.

x	S	x	S
.0001	.184	.9	.909
.001	.22	1.0	1.0
.01	.277	1.1	1.112
.1	.40	1.2	1.258
.3	.53	1.3	1.47
.5	.641	1.4	1.88
.7	.762	1.445	2.7183

Table 7

The highest value of x in this table is the eth root of e (*i.e.*, $e^{1/e}$), and this corresponds to $S = e(2.7183)$. S does diverge for $x > e^{1/e}$.

We can approach the problem analytically in the following way. Since $S = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}\rightarrow\infty$, we can raise x to the S power and get the equation $x^S = S$, since one more x in the infinite sequence does not make any difference. Taking logarithms of both sides, $S \log x = \log S$, or $\log x = (\log S)/S$. But now a curious thing occurs. If x is only slightly larger than 1, its log is very small, and $\log x$ can be small *either* because $\log S$ is small, *or* because S is large. For each value of $1 < x < e^{1/e}$ in table 7 a second value of S can be found, as is shown in table 8.

$x \rightarrow$	1.1	1.2	1.3	1.4
$S \rightarrow$	38.2	14.7	7.85	4.4

Table 8

This second value of S can only be computed by considering $\log x = (\log S)/S$. The smaller value can be computed directly by raising x to the appropriate power.

Wilkes-Gordon Construction

A farmer set out to erect a fence in a perfectly straight line. The only tool he had to help him locate the postholes was a straightedge. With it, he could continue any straight line indefinitely. He started to lay out the fence and suddenly found that the line ran smack into a large tree. He didn't want to start the line over again to avoid the tree, and he wanted to continue the line on the other side of the tree. How could he do this, using only a straightedge?

The solution of the problem can be obtained by the use of Desargues' Theorem,* or by a construction devised by R. Wilkes and proved by G. Gordon.† Desargues' Theorem is easy to prove in three dimensions, but difficult to prove in two dimensions. The two-dimensional theorem is required in this case.

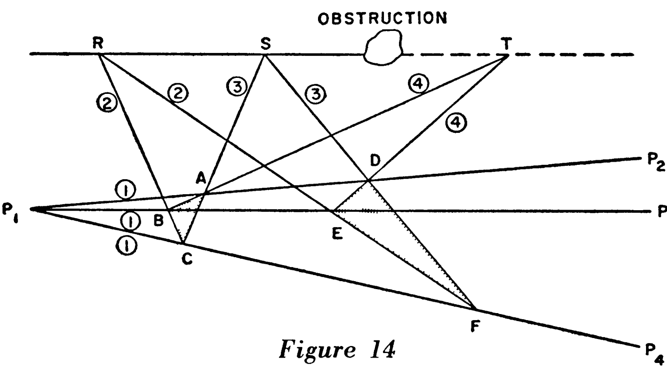


Figure 14

* Courant and Robbins, *What Is Mathematics?* New York: Oxford Univ. Press, 1941, p. 170.

† *Ibid.*, p. 197, problem 6, by Steiner, which is substantially the same as the Wilkes-Gordon Construction.

Desargues' Theorem in two dimensions can best be described by reference to figure 14. It states that if two triangles are drawn in a plane so that the three lines passing through pairs of corresponding vertices intersect at one point, then the extensions of corresponding sides of the triangles will intersect in points that are collinear. In figure 14 lines AD, BE, and CF intersect at a common point, P_1. The theorem then states that the intersections of BC with EF, AC with DF, and AB with DE, all intersect at points R, S, and T, which lie in a straight line. The problem can also be solved by the Wilkes-Gordon* Construction. The Wilkes-Gordon Construction can be described with the aid of figure 15.

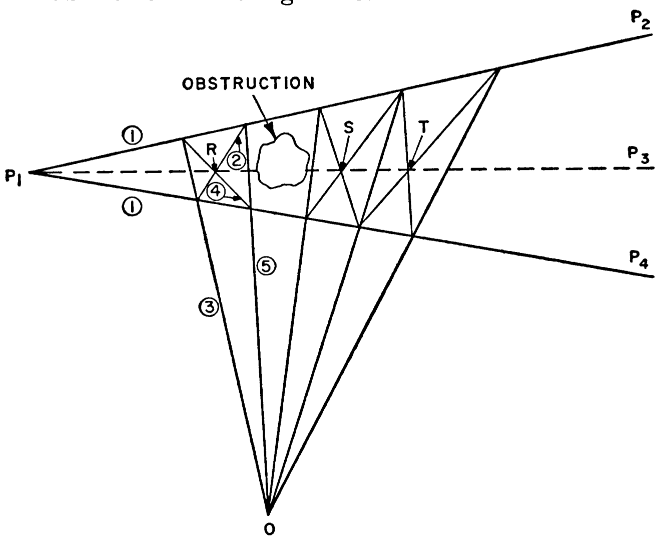


Figure 15

This construction states that if two lines from a common point, P_1, are intersected by a fan of lines from a

* Both Mr. Wilkes and Dr. Gordon are engineers at RCA.

second point, O, then the intersections of the diagonals of the quadrilaterals formed by these two sets of lines (R, S, and T) are all collinear with P_1.

To solve this problem using Desargues' Theorem, we refer to figure 14, and we choose R and S to be on the line before the obstruction. We choose a point P_1 arbitrarily, and draw lines P_1P_2, P_1P_3, and P_1P_4. From point R, lines RC and RF are drawn. From point S, lines SC and SF are drawn. These lines define points B, A, D, and E, as is shown in figure 14. Lines AB and ED intersect at a point T, on the same straight line as R and S. Then, R_1 and S_1 can be chosen (to be different from R and S, but on the same straight line). The entire procedure is repeated, and point T_1 is defined. T and T_1 are each collinear with R and S, and therefore define the line on the other side of the obstruction.

The Wilkes-Gordon Construction can be used as shown in figure 15. Point P_1 is chosen on the line which had been drawn. From point P_1, lines P_1P_2 and P_1P_4 are drawn. Then a diagonal is drawn through the arbitrary point R. (R is arbitrary except that it must be on the original line.) From the intersection of that diagonal with P_1P_4, a straight line is drawn. This straight line is labeled 3 in figure 15. Through point R and the intersection of line 3 with P_1P_2, a second diagonal is drawn. The intersections of the two diagonals with P_1P_2 and P_1P_4 define line 5. The intersection of line 3 with line 5 defines point O. Now, all lines from O which intersect P_1P_2 and P_1P_4 define quadrilaterals, and the intersections of the diagonals of these quadrilaterals (such as S and T) are collinear with P_1 and R.

Desargues' Theorem requires a new construction for each individual point. The Wilkes-Gordon Construction gives many points from one construction and is somewhat more economical.

Proof of the Pythagorean Theorem

The following proof of the Pythagorean Theorem has a certain attractiveness about it, primarily, I suppose, because it is so simple and candid. Consider the triangle to be investigated with sides a, b, and c.

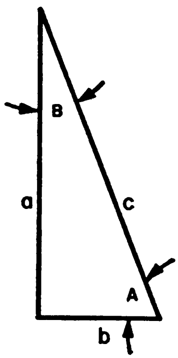


Figure 16

Let us now arrange four such triangles as shown in figure 17. Since angles A and B are complementary, each

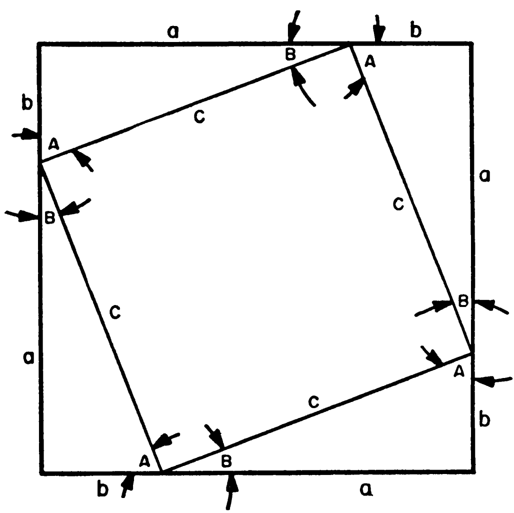


Figure 17

angle of the quadrilateral must be 90°. And since each side of the quadrilateral is c, the quadrilateral is a square. According to one reckoning, the entire area will be $(a + b)^2$, and according to another reckoning it will be c^2 and four triangles. The area of each triangle is $ab/2$ and the expression for the total area reads:

$$\begin{aligned}(a + b)^2 &= 4(ab/2) + c^2 \\ a^2 + 2ab + b^2 &= 2ab + c^2 \\ a^2 + b^2 &= c^2\end{aligned}$$

The Advantages of Compound Interest

In order to attract customers, banks often offer interest, compounded not once or twice, but *four* times a year. It requires but a few moments to calculate the actual benefits of such compound interest. If the bank offers interest at $i\%$, compounded once a year, then the money on deposit grows by a factor $(1 + i)$ each year. If the bank offers the $i\%$ interest, compounded n times a year, then we must divide the year into n parts, and apply a rate of interest (i/n) for each of those periods. Thus, the money would grow by a factor $[1 + (i/n)]$ each period or $[1 + (i/n)]^n$ each year, because each year has n periods. That is, the money invested at 3% compounded quarterly is worth, at the end of the year, $[1 + (.03/4)]^4$ times as much. In table 9 are listed the equivalent factors for different intervals of compounding the interest. Also, shown in this table are the factors applicable when the interest is compounded continuously.

The factor by which money grows if the interest has been compounded n times a year has been given as $[1 + (i/n)]^n$. As the interest is compounded more and more frequently, n approaches infinity, and the factor approaches the limit e^i. The expansion of e^i is:

$$e^i = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + \ldots + \frac{i^n}{n!} + \ldots$$

Table 9 lists the equivalent factors for several rates and different periods for compounding.

It can be seen from this table that the advantage of having the interest compounded quarterly may, in certain cases, be canceled by the cost of an additional postage stamp.

n	2%	3%	4%	5%
1	1.02000	1.03000	1.04000	1.05000
2	1.02010	1.03023	1.04040	1.05063
4	1.02015	1.03034	1.04060	1.05094
∞	1.02020	1.03045	1.04081	1.05127

Table 9

5: Coin Weighing Problems

Bachet's weight problem asks what set of weights is required to weigh all integral values of pounds (*i.e.*, 1, 2, 3, etc.) up to 31, if the weights can be placed on one side of the balance only. A simple extension asks for those weights required to weigh from 1 to 40, if one is allowed to put weights on either side of the balance. In the first case the answer is the progression of powers of 2 (the binary sequence), namely, 1, 2, 4, and 8, etc., and in the second case, a progression of powers of 3, namely, 1, 3, 9, and 27, etc. Another modification asks what weights are required to weigh up to 12 pounds, if one has a four-pan balance. The pans are symmetrical on opposite sides of the fulcrum, and the outer pans have twice the lever arm of the inner ones. The answers in this case are 1 and 5. (The general term is 5^n.)

Determination of Odd Coin Among Twelve Coins

A more sophisticated problem is the following: You are given a pile of twelve coins, all of which look the same, and are of the same denomination, and are told that one, and only one, of these coins is counterfeit. But you do not know whether the counterfeit coin is heavier or lighter than a true coin. Using an equal-arm balance, you are

asked to identify the false coin in three weighings, and to state whether it is heavier or lighter than the rest.

A solution to this problem requires the recognition of two facts: (*a*) if equal numbers of coins are placed in the two pans, and they balance, then the false coin *must* be in the group of coins which have not yet been used, and (*b*) if two groups of coins do not balance, then each coin in the pan that goes down is potentially heavy (*i.e.*, *if* the false coin is heavy, it must be one of those, and each coin in the pan that goes up is potentially light.)

It is convenient to label potentially heavy coins H, and potentially light coins L. The original pile of 12 coins is divided into 3 groups of four each, and two of these piles are compared. If the two piles balance, we know the false coin is in the last pile of four. Means for selecting the false coin from those four will become apparent in the following discussion. If the two original piles did not balance, each of the coins in the heavier pan is labeled H, and each one in the lighter pan is labeled L. The four coins in the last group are labeled G (for good) because there is only one false coin, and the last pile must accordingly be all good.

Then we put 2 H and 2 L in the left pan for the second weighing, and we put 1 L, 1 H, and 2 G in the right pan. If the left pan goes down, then either one of the 2 H in the left pan was actually heavy or the L in the right pan was actually light. Then, weighing one of the 2 H and the L against 2 G identifies the culprit. If they balance, the remaining H must be the false coin. If they do not, the H or the L is at fault and can be weighed against the G.

If the first weighing balanced, we would then take two of the remaining four, and weigh them against one of those remaining four plus a good coin (taken from the eight which must be all good). Again, 2 H and 1 L (or its image) are determined, or a single unknown coin is left. In general, $(3^N - 3)/2$ coins (where N is the number of weighings) can be weighed in this way.

Weak and Strong Solutions

The previous solution is known as a weak solution because each arrangement of coins to be weighed depends on the result of the previous weighing. "Strong" solutions are those where the arrangements to be weighed are specified in advance. (The words weak and strong are not intended to reflect on the intellectual prowess of the problem solver. They are simply a convenient definition.) It is quite reasonable that a strong solution for this problem should exist. There are three possibilities for each weighing—the left pan can go down (S), the right pan can go down (D), or they can balance (B). Since three weighings are performed, there are 3^3 or 27 different possibilities. One of these, the one where the pans balance each time, gives us no information on whether the suspected coin is heavy or light, and it must be discarded. The remaining twenty-six possibilities can be arranged into thirteen pairs, in which each arrangement of D's and S's and B's has its mirror image (where the D's and S's are interchanged). The use of only one of each of these symmetrical pairs enables us to determine whether it is *this* coin that is heavy, or *that* coin that is light.

A Strong Solution

One of the easier ways to construct a strong solution is to start by preparing a list of all groups of 3 D's, S's, or

1	S	S	B	8	B	D	B
2	S	B	S	9	B	B	D
3	B	S	S	10	S	D	D
4	S	D	B	11	D	S	D
5	D	B	S	12	D	D	S
6	B	S	D	13	S	S	S
7	D	B	B				

Table 10

B's in which no group is the mirror image of any other. Such a listing is shown in table 10.

From table 10, we can go on to derive the strong solution as is shown in table 11:

	Left Pan (S)	*Right Pan* (D)	*Not Used* (B)
1st weighing:	1, 2, 4, 10	5, 7, 11, 12	3, 6, 8, 9
2nd weighing:	1, 3, 6, 11	4, 8, 10, 12	2, 5, 7, 9
3rd weighing:	2, 3, 5, 12	6, 9, 10, 11	1, 4, 7, 8

Table 11

This table is constructed from table 10, by assuming that each coin in turn is heavy, and assigning it to the left, right, or unused pile for each of the three weighings in accordance with table 10. The possibility of its being light is automatically provided for, because no mirror images were used. A restriction, of course, is that the number of coins on the left pan must equal that on the right pan. This restriction prevents us from using the thirteenth possibility in table 10, namely SSS. Any other one of the thirteen listings could have been omitted instead of that one. The odd coin is determined by finding that line of table 10 which corresponds to the actual happenings in the three successive weighings.

The problem of the twelve coins can be modified to use the thirteenth possibility, as shown in table 12. To do this, consider a group of fourteen coins in which one coin is pointed out as a true coin, and in which a false coin is somewhere in the remaining thirteen. It is not known

	Left Pan	*Right Pan*	*Not Used*
1st weighing	1, 2, 4, 10, 13	5, 7, 11, 12, G	3, 6, 8, 9
2nd weighing	1, 3, 6, 11, 13	4, 8, 10, 12, G	2, 5, 7, 9
3rd weighing	2, 3, 5, 12, 13	6, 9, 10, 11, G	1, 4, 7, 8

Table 12

whether the false coin is heavier or lighter than the rest. Table 12 shows a strong solution in which the thirteen coins are numbered, and the fourteenth is labeled G.

This procedure will identify the false coin among thirteen coins, and will indicate whether it is heavy or light. A weak solution is also applicable to this problem.

Strong solutions are more difficult to prepare than weak solutions, and this accounts for the fact that such solutions are far less common. A weak solution for the problem above with 120 coins and five weighings can easily be done. A strong solution for this case is long and arduous. In general, the odd coin from among $(3^n - 3)/2$ coins can be determined in n weighings, and it can be determined whether that coin is heavy or light.

It is reported by Gnedenko,* that the Russian Mathematical Olympiads asked for the determination in four weighings, of a single light coin among 80 coins. If it is known that the coin is light (or heavy) then one of 3^N coins can be determined in N weighings. In the present case, using four weighings, one out of $3^4 = 81$ coins can be determined. It may be that the number 80 (rather than 81) was used because 80 is not divisible by 3, and may be pedagogically more appealing.

Another Coin Problem

Part of the definition of an interesting problem is that it seems offhand to be impossible to solve. Accordingly, the following is a very interesting problem.

In a certain country, there are N (an arbitrary number) denominations of coins in use. Each denomination of coin is made by one and only one company and the proper weight of each kind of coin is specified by the legislature.

* Gnedenko, B. V., "Mathematical Education in the USSR," *American Mathematical Monthly, 64* p. 389 (1957), Problem 3, p. 397.

One of the companies is suspected of cheating, but it is not known whether it cheats by making the coins somewhat light (and pocketing the difference) or by using a heavy base metal in the center of the coin, thus making it heavier than it should be. Armed only with a list of the proper weights and a spring balance, and having available as many of each coin as we wish, determine in two weighings which company is cheating and determine the discrepancy. (It is assumed that only one company is cheating.)

This problem, at first glance, seems impossible. At least one person prepared a "proof" of impossibility when the problem was first proposed to him. Needless to say, there was a flaw in his "proof." To solve this problem, weigh a group consisting of one coin of each type. Knowing this total weight and also the sum of the legal weights, determine the discrepancy, of the suspected coin. You still won't know which coin is off by this amount. For the next weighing, we take one of the first coin, two of the second, three of the third, etc., and again find the discrepancy. The discrepancy this time will be md where d is the previously determined mismatch, and m is the rank of the coin.

6: Fermat's Theorem; Sailors and Coconuts

One of the leading figures of classical number theory was a Frenchman, Pierre Fermat. Fermat was a magistrate most of his life, but nonetheless he made some of the weightiest contributions to number theory. His theorems can be proved using nothing more complicated than high-school algebra. Many people feel that Fermat's proofs may have been done the same way. This statement is more interesting in view of the following story:

Fermat was in the habit of merely announcing certain theorems without presenting a proof. All but one of his theorems have since been verified. The big exception is his statement that $2^{2^n} + 1$ is always a prime number (has no factors besides unity and itself), but he was unable to prove it. Euler proved the statement false by showing that $2^{2^5} + 1$ is divisible by 641. (In fact, no primes have been discovered among the Fermat numbers, $2^{2^n} + 1$, for any n greater than 4.)

How, then, are we to regard Fermat's notation on the margin of his copy of Diophantus that he had found "a most elegant proof of the statement that $x^n + y^n = z^n$ has no solution in integers for $n \geq 3$. But this margin is too small to contain the proof." Mathematicians have been battling with this problem ever since, and have still not found a proof.

The Fermat "blooper" raises a very interesting

question, namely, how can one determine whether a number is prime. Fermat, as well as other number theorists, was deeply concerned with this problem. The mathematicians of Fermat's time may have had a slight advantage because they apparently had wonderful ways of recognizing prime numbers. Such recognition seems uncanny today.

For example, Fermat was asked whether or not the number 100,895,598,169 was prime. Fermat is said to have answered instantly that the number is not prime, but that it is the product of 898,423 and 112,303, and further that both of these were prime.

The field of number theory interests not only the hard core of number theorists, but also a soft outer shell of gifted amateurs. Wilson was one such amateur. He derived an expression for n which, if satisfied, is both necessary *and* sufficient for n to be a prime number. Wilson's theorem states that if $(n - 1)!$* $+ 1$ is evenly divisible by n, then, and only then, is n a prime number. This is certainly a beautifully simple condition for necessity and sufficiency, but the difficulty in calculating $(n - 1)!$, even when n is small, limits the usefulness of the theorem.

In order to understand the Fermat theorem with which this chapter is concerned, we must become familiar with two new notions. The first is the notion of modulus. In using this concept, we say that two numbers are congruent modulo N if the remainders, upon dividing each number by N, are equal. For example, $7 \equiv 2 \pmod 5$ [read: 7 is congruent to 2 (modulo 5)], or $11 \equiv 2 \pmod 3$. (This simply means that both 7 and 2, when divided by 5, have the same remainder—2. Likewise, 11 and 2 when divided by 3 both have a remainder of 2.) We use this technique every day when we speak of the time of the day or the day of the week, since we subconsciously subtract an

* $(n - 1)!$ (where $n = 6$) would be $5!$ or $5 \times 4 \times 3 \times 2 \times 1$.

integral number of days (and hours) in the first case, and an integral number of weeks in the second.

The second new idea is that of the Euler ϕ function: $\phi(n)$ is defined as the number of integers (including 1), which are smaller than n and are prime to n (*i.e.*, have no common factors with n). Thus, $\phi(6) = 2$ because only 1 and 5 are less than 6 and prime to it. Also $\phi(7) = 6$ and $\phi(8) = 4$. Fermat's theorem states that $a^{\phi(n)} \equiv 1 \pmod{n}$ if a is prime to n.

Considering that $\phi(n)$ appears to be such an unwieldy function, it is surprising that such a simple relationship exists. $\phi(3) = 2$ because only 1 and 2 are less than 3, and prime to it. Thus, Fermat's theorem states that:

$$a^{\phi(3)} \equiv 1 \pmod{3} \text{ or } a^2 \equiv 1 \pmod{3}$$

This states that the square of any number which is not a multiple of 3 has a remainder of one when divided by 3. *E.g.*, $5^2 = 25 \equiv 1 \pmod{3}$, $7^2 = 49 \equiv 1 \pmod{3}$. Please note, however, that $6^2 = 36 \equiv 0 \pmod{3}$. The restriction that a be not divisible by n is quite apparent. Similarly, the fourth power of any number not divisible by 5 has a remainder 1 when divided by 5. Thus, $2^4 = 16 \equiv 1 \pmod{5}$, $3^4 = 81 \equiv 1 \pmod{5}$. But $5^4 = 625 \equiv 0 \pmod{5}$. A proof of this theorem is given at the end of this chapter.

Fermat's theorem is the key to the following problem: What is the smallest number such that moving the right-hand digit to the front of the number (and shifting all the other digits one place to the right) merely doubles the original number? This problem can be attacked by the "hammer and tongs" approach, in which we say the end number must be at least 2 because the original number must start with at least 1. Then, working to the left, the successive numbers must be 4, then 8, then 6 (but carry one), then 3 (carry one), etc. We stop when we come to a zero or a 1, preceded by a number less than 5. (If it were greater than 4, double the number would start with a 3.)

But there is little doubt that this method is hardly elegant, and surely possesses no generality. A much more satisfying solution is the following one:

Let the number we seek, N, be represented by:

$$N = a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10^1 + a_0 10^0 \qquad (1)$$

We can also write:

$$2N = a_0 10^n + a_n 10^{n-1} + \ldots + a_2 10^1 + a_1 10^0 \qquad (2)$$

Multiply (2) by 10, getting:

$$20N = a_0 10^{n+1} + a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10^1 \qquad (3)$$

Subtract (1) from (3):

$$19N = a_0(10^{n+1} - 1); \; N = a_0\left(\frac{10^{n+1} - 1}{19}\right)$$

Fermat's theorem states that the factor in parentheses is an integer when $n + 1 = 18$ (because $\phi[19] = 18$). The hammer-and-tongs solution corresponds to the case for $a_0 = 2$. The case for $a_0 = 1$ is a solution only if we agree to accept the first digit in the quotient as really a 0. Thus: $10^{18} - 1 = 999999999999999999$; $(10^{18} - 1)/19$ is shown in figure 18.

This number is a solution for all except the purist who insists that zeros to the left of the first significant digit are not part of the number and should be neglected. For him, the solution is double this number.

The original question could have asked for the altered number to be 3 times as large as the first number. In this case, the number would have come out to be a multiple of $(10^{n+1} - 1)/29$. Fermat's theorem requires $n + 1 = 28$ in this case. Such large numbers, however, are cumbersome and hardly suitable for chats about "nice little problems." Sometimes the problem is varied so that moving the first digit to the right side increases the original number by fifty percent.

```
   052631578947368421
   ------------------
19)999999999999999999
   95
   --
    49
    38
    --
    119
    114
    ---
      59
      57
      --
       29
       19
       --
       109
        95
       ---
        149
        133
        ---
         169
         152
         ---
          179
          171
          ---
            89
            76
            --
            139
            133
            ---
              69
              57
              --
              129
              114
              ---
               159
               152
               ---
                 79
                 76
                 --
                  39
                  38
                  --
                   19
                   19
                   --
                    0
```

Figure 18

Three Men and Twelve Lumps of Sugar

The study of number theory can be fascinating. Its application ranges from exceedingly simple subjects to extremely abstruse ones. At the complicated end of the spectrum is Kummer's discovery that no integer n which divides a numerator of a specific set of Bernoulli numbers* can possibly satisfy Fermat's equation $x^n + y^n = z^n$ in integers. The other end of the scale is indicated by the following story:

Three men went into a diner, and each ordered a cup of coffee. The waitress brought the three cups of coffee and a dish with twelve lumps of sugar. Each man took an odd number of lumps of sugar, and when they had finished, there was no sugar left. How many lumps did each man take?

It requires only a few moments to recognize that the sum of three odd numbers must be odd itself. So there must be a trick somewhere, and there is.

The first man took one lump, the second man took one lump, and the third man took ten lumps. "Aha!" you will cry, "ten is not an odd number!" And then, we slyly inquire, "Do *you* know anyone who takes ten lumps of sugar in *his* coffee?"

Perhaps a better example of simple number theory is the problem asking for a proof that all primes are of the form $6k \pm 1$. "Forsooth!" the reader may say. "This chap has, indeed, lost his marbles. There is no formula to give only primes." Please note that it states that all primes *must* be of this form—*not* that all numbers of this form are prime.

To prove this, we note that $6k + 2$ can be factored into $2(3k + 1)$, $6k + 3$ into $3(2k + 1)$, $6k + 4$ into

* The Bernoulli numbers B_n are those which satisfy the expression

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \times \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} \equiv 1$$

$2(3k + 2)$, and $6k + 5$ is equivalent to $6k' - 1$ (where k' is simply one larger than k). So all numbers which are not $6k \pm 1$ are composite, and the primes, then, must exist among the $6k \pm 1$.

Sailors and Coconuts

We now move on to the problem of the five shipwrecked sailors who collect a pile of coconuts and decide to divide it the following day. In the middle of the night, one of the sailors wakes up. To make sure he isn't cheated the next day, he takes his fifth of the coconuts right then. He counts them, and finding one too many for the total number to be divisible by five, throws it away, takes his share, and goes back to sleep. Each of the others goes through the same process, sequentially, of waking up, finding one too many for the number to be divisible by five, throwing the extra away, and taking his fifth. What is the smallest number of coconuts that the original pile might have contained? The modulus technique can be used to solve this problem but it is fairly cumbersome. Actually, the following solution, attributed to E. Fermi,* has more charm and appeal.

This solution says minus four is surely a mathematical solution, because each man, seeing minus four coconuts, throws one away, leaving minus five, and takes his fifth—leaving minus four again. To get a real and positive solution, however, we must add the smallest number divisible by 5, 5 times, which is 5^5. So the answer is:

$$5^5 - 4 = 3{,}125 - 4 = 3{,}121$$

An interesting note might be added here. Several years ago, a problem appeared in the "Letters to the Editor" column of *Life* magazine. This problem asked for

* The truth of this story is immaterial. It indicates the high regard that the physics community had for Fermi.

the solution of the aforementioned problem for three sailors, each throwing one coconut away before taking his third, but with the added condition that the next morning, a fourth coconut had to be thrown away in order to divide the pile into thirds. The solution to this problem is "obviously" $3^4 - 2$, or 79, if we are aware of the solution attributed to Fermi. Two weeks after the problem appeared all sorts of people had written in their answers. A banker said he did it in two nights, a lawyer had done it in three hours, etc. A mathematics student from M.I.T. bragged that he had done it in twelve minutes! The answer can be written down instantly—*if* we only know how!

The proof that $n^n - (n - 1)$ is a solution for the problem of the sailors and coconuts is fairly straightforward. We start with

$$[n^n - (n - 1)] \text{ coconuts}$$

After throwing one away, and taking one nth, there are left:

$$\frac{n-1}{n}[n^n - n] = [(n - 1)n^{(n-1)} - (n - 1)]$$

And after the second person throws one away, and takes an nth, there are left:

$$\frac{n-1}{n}[(n - 1)n^{(n-1)} - n)] = (n - 1)^2 n^{(n-2)} - (n - 1)$$

This process can not be continued after the nth removal of one nth of the objects, because the term which started out as n^n will have been metamorphosed into $(n - 1)^n n^0$. If it were desired that the process be performed r times, we would have had to start with a number of objects $n^r - (n - 1)$.

In all the versions of this problem that I have seen, the number of people has been prime. This restriction, however, is required only if we wish the number of coco-

nuts, on the next morning to be evenly divisible by the number of people. (Fermat's theorem tells us that $(n-1)^n - (n-1) = (n-1)[(n-1)^{n-1} - 1]$ is always divisible by n, if n is a prime number.)

For example, if the problem of the sailors and coconuts involved only four sailors, each taking his fourth (after throwing one away), the number of coconuts to start with would have been $4^4 - 3 = 256 - 3 = 253$.

Table 13 gives values of N (the total number of objects) for different values of n (the total number of people) and shows the amount left by each person at each step in the process. It is seen that the number of objects left by the last person is divisible by the total number of people only when n is a prime.

n	*Total number of coconuts*	*1st person leaves*	*2nd person leaves*	*3rd person leaves*	*4th person leaves*	*5th person leaves*	*6th person leaves*
2	3	1	0				
3	25	16	10	6			
4	253	189	141	105	78		
5	3,121	2,496	1,996	1,596	1,276	1,020	
6	46,651	38,875	32,395	26,995	22,495	18,745	15,620

Table 13

$a^5 - a$ Is Divisible by 30

Another problem which yields very neatly to Fermat's theorem asks for a proof that all numbers of the form $a^5 - a$ are evenly divisible by 30. To prove this, we first factor an a out of the above expression, getting $a(a^4 - 1)$. Fermat's theorem states that $a^4 - 1$ is always divisible by 5, unless a itself is a multiple of 5. So $a(a^4 - 1)$ is always divisible by 5. Then $a^4 - 1$ is factored, and the original number becomes $a(a^2 + 1)(a^2 - 1)$. By an argument

similar to the above, $a(a^2 - 1)$ is always divisible by 3. Finally $a^2 - 1$ is factored into $(a + 1)(a - 1)$, and one of the two numbers a or $a + 1$ must be divisible by 2. We have thus shown that $(a^5 - a)$ is always divisible by 2, 3, and 5, the product of which is 30.

Divisibility of $a^{13} - a$

A problem which is similar to the above, but which is more extensive, is the one asking for a proof that 2,730 is a factor of any $a^{13} - a$. The prime factors of 2,730 are 2, 3, 5, 7, and 13. We must show divisibility by each of these numbers.

Fermat's theorem tells us that 13 is a factor of any $a^{13} - a$. $a^{13} - a$ can be factored into $a(a^6 + 1)(a^6 - 1)$. Seven is a factor of any $a(a^6 - 1)$. The $(a^6 - 1)$ can be factored into $(a^3 + 1)(a^3 - 1)$. The $(a^3 + 1)$ can be factored into $(a + 1)(a^2 - a + 1)$ and the $(a^3 - 1)$ into $(a - 1)(a^2 + a + 1)$. The entire number has been factored thus:

$$(a^{13} - a) = a(a^6 + 1)(a + 1)(a^2 - a + 1)(a - 1)(a^2 + a + 1)$$

One of three numbers in a row is divisible by 3, and one of two numbers in a row is divisible by 2. Therefore, the product $(a - 1)a(a + 1)$ must be divisible by 2 and 3. We have shown divisibility by 2, 3, 7, and 13, and only 5 remains to be shown as a divisor.

We must now change tactics and write a as $(10k + r)$. (This is true for any a we choose.) If the r happens to be 0 or 5, a is surely divisible by 5. If r is 1 or 6, then the $(a - 1)$ term would be divisible by 5, and similarly, if r is 4 or 9, the $(a + 1)$ term is divisible by 5. If r is equal to 2, then the $(a^6 + 1)$ would be divisible by 5—because each member of the expansion of $(10k + 2)^6 + 1$ is divisible by 10, except for the term $2^6 + 1 = 64 + 1 = 65$, which is again divisible by 5. By a similar argument, if $r = 3$ or 7 or 8,

each number in the expansion would be divisible by 10 (or 5), except for the 3 or 7 or 8 to the sixth power plus 1. But 3 or 7 or 8 to the sixth power always ends in a 4 or a 9, and therefore $(a^6 + 1)$ would again be divisible by 5. The known factors of $a^{13} - a$ now include 5, and are thus 13, 7, 5, 3, and 2, and the product of all these factors is 2,730.

Proof of Fermat's Theorem

Fermat's theorem states that $a^{\phi(n)} \equiv 1 \pmod{n}$ where a is prime to n and $\phi(n)$ has been defined as the number of numbers smaller than n and prime to it. Let these numbers be represented by:

$$s_1,\ s_2,\ s_3,\ \ldots\ s_{\phi(n)}$$

There can be only $\phi(n)$ different ones. Multiply each of these numbers by a and write each of these products as an integral number of n's plus some remainder, R. Thus:

$$as_1 = Q_1 n + R_1;\ as_2 = Q_2 n + R_2; \ldots as_{\phi(n)} = Q_{\phi(n)} n + R_{\phi(n)} \quad (1)$$

There are $\phi(n)$ such equations because there are $\phi(n)$ different s's. The Q's above are not necessarily different from each other. Some may even be zero. Each of the R's must be prime to n. If this were not so, we could factor out the common factor from it and n on the right, and either a or s would then have to be divisible by this factor of n. But this is contrary to our assumptions. Each R must also be different from any other R. For if any two R's—say, R_x and R_y — were equal, we could subtract those two equations from each other and get:

$$as_x - as_y = (Q_x n - Q_y n) + R_x - R_y$$

Since $R_x = R_y$ we can factor and write:

$$a(s_x - s_y) = (Q_x - Q_y)n$$

The Fundamental Theorem of arithmetic tells us that if n is a factor on the right side of the equation, then it must also be a factor on the left side. But all s's are less than n and prime to it, and a is prime to n. So, n cannot be a factor of the left side of the equation. This impasse can only mean that no $R_x = R_y$. Since the R's are all smaller than n, and prime to n, and different from each other, and there are $\phi(n)$ of them, they must be all the s's in some different order.

Let us now multiply all the equations in (1) together. We get:

$$a^{\phi(n)}[s_1 \times s_2 \times s_3 \times \ldots \times s_{\phi(n)}] = Ln + [R_1 \times R_2 \times R_3 \times \ldots \times R_{\phi(n)}]$$

L represents some large number whose actual form does not interest us. The product in square brackets on the right must be the same as that in square brackets on the left. We can thus factor and write

$$(a^{\phi(n)} - 1)[s_1 \times s_2 \times \ldots \times s_{\phi(n)}] = Ln$$

Since all the s's are prime to n, their product is also, and it must then be $a^{\phi(n)} - 1$ which is divisible by n. Thus:

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

7: Partitioning Problems*

The Maid at the Stream

A maid is sent to a stream to get exactly two quarts of water, but she has only a 7-quart jug and an 11-quart jug. How can she measure the desired amount?

The Euclidean Algorithm, which is described at the end of this chapter, yields integers r and s which satisfy the equation:

$$rp - sq = 1 \tag{1}$$

where p and q have no common factor, *i.e.*, are relatively prime integers. For the purposes of this problem, this equation can be interpreted as saying that if a p-unit container is filled exactly r times, and is emptied only into a q-unit container, and this q-unit container is emptied exactly s times (into the master container), then exactly one unit must remain in the p-unit container.

If the desired amount were not a single unit, but a units of the substance, then each term in equation (1) could be multiplied by a, giving: $arp - asq = a$.

This equation indicates a way of obtaining a units, but it might not indicate the procedure requiring the minimum number of transfer operations. In such a case, two things can be done. If, in the equation above, $ar > q$

*This chapter is an expansion of "A Solution for Certain Types of Partitioning Problems" by M. H. Greenblatt, *The Mathematics Teacher*, vol. 54, no. 7, p. 556.

and $as > p$, then some multiple of pq can and should be subtracted from each term on the left. This altered equation will then indicate a more economical solution of the problem. Also, the equation:

$$r'p - s'q = -1 \tag{2}$$

which can be obtained by judicious manipulation of (1), should be investigated. In this case, in which r' and s' are different from r and s, the roles of the p and q containers are reversed; the q container is filled s' times and the p container is emptied r' times. Again, in this case, the terms on the left in equation (2) should be decreased by a multiple of pq if possible. Either this solution or the one derived from equation (1) will be the most economical solution. Equation (1) and equation (2) may lead to the same solution, and then both are equally economical.

These ideas can be clearly illustrated by solving an actual problem. Suppose, for example, that a maid must get exactly two quarts from a stream, and she has only a 7-quart jug and an 11-quart jug available to her. From the Euclidean Algorithm we find $(2 \times 11) - (3 \times 7) = 1$. As shown above, this can be changed to $(4 \times 11) - (6 \times 7) = 2$ and this equation states that at least $4 + 6$ filling-and-emptying operations are required to get two quarts if one starts by filling the 11-quart jug. To investigate the alternate solution, we proceed as follows:

	$(2 \times 11) - (3 \times 7) = 1$
Multiply by 6:	$(12 \times 11) - (18 \times 7) = 6$
Subtract 7:	$(12 \times 11) - (19 \times 7) = 6 - 7 = -1$
Subtract 7×11:	$(5 \times 11) - (8 \times 7) = -1$
Multiply by 2:	$(10 \times 11) - (16 \times 7) = -2$
Subtract 7×11:	$(3 \times 11) - (5 \times 7) = -2$

According to this solution, $3 + 5$ filling and emptying operations are required to get two quarts if one starts by filling the 7-quart jug. This second solution is hence the

more economical one for this problem. The total number of transfer operations is twice the number of filling and emptying operations minus one. This is so because each filling operation must be followed by a transfer to the other container, and each emptying operation, except the last, must be followed by a transfer from the other container.

The question of whether the source is very large (as in the case of the stream) or very small is not important because the size of the source does not enter into the solution. The only requirement is that the source capacity be at least $(p + q - 1)$. If the size of the source is less than this, certain mathematical solutions are not physically realizable. If the numbers p and q are not relatively prime, equation (1) is no longer true, certain numbers can then not be formed by multiplication by a, and the problem loses some of its interest.

Division of Balsam Among Three People

I once came across a variation of this problem in which three men had stolen a jar containing 24 ounces of balsam and wanted to divide it equally among themselves. They found a 5-ounce container, a 9-ounce container, and a 10-ounce container. (How improbable can these problems get, what with balsam and such unusual size containers?) This problem is subject to the same analysis as has been given, but there is at least one pitfall which must be avoided. It can, in a sense, be a much tougher problem than the one we have considered.

Most people can see that 9 ounces can be put into the 10-ounce container, and then the 10-ounce container filled from the refilled 9-ounce container leaving the required 8 ounces in the 9-ounce container. But at this point, the 8 ounces in the 9-ounce container *must* be transferred to the 10-ounce container. Otherwise, since 5 and 10 are

not relatively prime, we will not be able to form the second 8 ounces. However, if the 8 ounces is put into the 10-ounce container, then we have a 5-ounce and 9-ounce container to operate with, and since 5 and 9 *are* relatively prime, we *can* form the second 8-ounce portion. The steps are: after pouring everything but the 8 ounces in the 10-ounce container back into the master container, that is 16 ounces, we fill the 9-ounce container, and from that fill the 5-ounce container, leaving 4 ounces in the 9-ounce container. These 4 ounces are put into the 5-ounce container, and from the refilled 9-ounce container, only one ounce can be poured into the 5-ounce container, which already has 4 ounces in it. Thus we have 8 ounces in the 10-ounce container, 8 ounces in the 9-ounce container, and there must then be 8 ounces between the 5-ounce container, and the master container. The division is completed.

In another variation, the three men would have 5-, 7-, and 11-ounce containers, and since these three numbers are all relatively prime, it can be solved by the methods previously outlined.

Another Urn Problem

Another man had a 5-unit jug, a 9-unit jug and 14 units of a substance which he wanted to divide into two equal portions. The Euclidean Algorithm tells us that:

$$(3 \times 9) - (4 \times 5) = 7 \quad (3 + 4 = 7)$$

and that: $(5 \times 5) - (2 \times 9) = 7 \quad (5 + 2 = 7)$

We can produce 7 units by filling either the 5-unit container, or the 9-unit container first. The total number of operations in both cases will be:

$$(2 \times 7) - 1 = 13$$

Additional problems of this sort can easily be devised.

Pile of Bricks

The same methods apply to solving the one about the man who had a pile of bricks. The number of bricks was such that when he tried to build a wall 2 bricks high there was one left over. When he tried to build it 3 bricks high, there was still one left over, and so on, for 4, 5, and 6 bricks in height. But when he tried to build the wall 7 bricks high, he had no bricks left over. What was the minimum number of bricks in the pile?

We can solve this problem by simply trying each number from seven on up until we finally discover that 301 has the desired properties. In addition to being time-consuming *and* inglorious, this method possesses little generality. A much better solution is the following:

The least common multiple of 2, 3, 4, 5, and 6 (that is, the smallest number evenly divisible by each of these) is 60, where all the prime factors of each number appear just once. Any multiple of 60, plus 1, will leave a remainder of one when divided by 2, 3, 4, 5, or 6. As n is varied, the numbers $60n + 1$ will be congruent to each of the integers from 0 to 6 (mod 7). We seek the lowest one congruent to 0 (mod 7), and in this case, $(5 \times 60) + 1 \equiv 0 \pmod{7}$. Again, 301 is the answer.

The multiple of 60 can be determined by trying each integer from 1 to 7, or it can be derived from the Euclidean Algorithm by finding the solution of the equation:

$$7r - 60s = 1$$

The number of bricks by which the pile *is* divisible should be a prime number. Otherwise a solution may not exist.

Poker Chips

A variation of this problem concerns a pile of poker chips. When it is divided into piles of two chips each,

there is one chip left over. When it is divided into piles of three, there are two left over, in piles of four, there are three left over, and so on for five, six, seven, eight, nine, and ten. But when the chips are divided into groups of eleven, there is none left over. What is the minimum number of chips in the pile?

As before, we form the number $2 \times 3 \times 2 \times 5 \times 7 \times 2 \times 3 = 2{,}520$, which is the smallest number divisible by each of the integers 2 through 10. All numbers of the form $2{,}520n - 1$ lack one unit to be divisible by the numbers 2 through 10. We must find that number of the family which is divisible by 11. The first number we try happens to have the desired property, because $2{,}519 \div 11$ equals 229.

Finding Any Rational Fraction of a Volume V

The milkmaid in the first problem had a fairly difficult assignment, but consider the plight of the maid in the present problem. She is given a glass jar of irregular cross section and of volume V, an earthenware jug of volume somewhat greater than V, a marking pencil, an infinite source of water, and a cap for the glass jar. She is required to produce any rational fraction of the volume V. (Most milkmaids would quit at this point.)

The solution consists of two parts. First, we show how the volume V can be divided into halves, quarters, eighths, etc., and then we show how any rational fraction can be represented in binary form (*i.e.*, how many halves, quarters, eighths, etc., must be added to equal any given fraction).

First, we put in an amount of water which we think is $V/2$. Then, with the marking pencil, we put a mark at that level of the liquid, put the cap on the jar, and turn it upside down. If the jar was half full of liquid, the level of the liquid will be exactly at the mark that we made.

If there is too much liquid, the inverted level will be above the mark; if there is too little liquid, the inverted level will be below the mark. In either case, we can add or subtract liquid and, by repeated trials, approach $V/2$ as closely as we wish. Finally, all the incorrect marks can be erased, leaving only the correct one.

Next, the volume $V/2$ is poured into the earthenware jug, and a volume which we guess to be $V/4$ is put into the jar. Again, we put a mark at the guessed $V/4$ level, add the $V/2$ from the jug, cap the jar, and turn it upside down. The level now should be up to the $V/4$ mark. In the same way as before, $V/4$ can be approached as closely as we wish. This procedure can then be repeated with the successive reciprocal powers of 2. Eventually, each fraction of the form $V/2^n$ and the sum of any of these fractions can be marked.

Any rational fraction can be expressed as a repeating decimal. This repeating decimal can be expressed in binary form in the following way:

The decimal is written out and multiplied by 2. If the product is less than one, then we write down a zero as part of the binary representation. If the product is greater than one we write a one for part of the binary

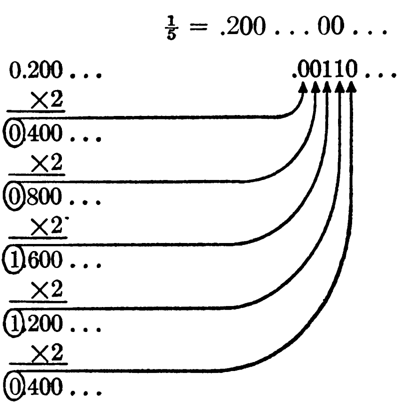


Figure 19

representation. Then the decimal part of the product is multiplied by 2 again, and whenever the product in decimal form is greater than one, we write a 1 for the binary representation, and simply neglect the one in the decimal for subsequent multiplications. The procedure thus consists of a successive multiplication of the decimal by 2, and the assignation of a zero or a one for the binary expression. Figure 19 shows how the fraction $\frac{1}{5}$ can be expressed in binary form. Having the binary expression, one can then pour from the jar into the earthenware jug the required fractions $V/2^n$.

In a variation of this problem, the volume V in the previous problem can be specified by a mark on the irregularly shaped jar, and a cap for the jar is *not* supplied. Again, any rational fraction of V is requested. In this case, the volume V of water is poured into the glass jar. Then water is poured from the jar into the jug until we guess that we have $V/2$ left. At this point, we mark the jar, pour the water in the jar out, and pour the water from the jug back to the jar. If the guessed volume were less than $V/2$, then the level of water from the jug would be higher and vice versa. If the levels were different, then we would take as our next guess a level between the two (for $V/2$ is greater than the lower mark, and less than the upper mark).

To determine $V/4$, we pour from V (into the jug) until we have what we think is $V/4$ left in the jar. One marks this level of liquid. We can then empty this water, pour $V/2$ from the jug into the jar (since $V/2$ has been marked on the jar) and empty that, and put the remainder from the jug into the jar. If the mark is higher than the level of the water, the mark is more than $V/4$, and vice versa. In any case, the various $V/2^n$ and combinations thereof can be marked. From this point on, the solution is the same as the previous one. (The milkmaid has just shot herself!)

The Euclidean Algorithm

The Euclidean Algorithm provides a simple way to find the greatest common denominator (GCD) of any two numbers. If the two numbers are relatively prime, (p and q) then a very modest amount of additional work provides the solution to $rp - sq = 1$. The Euclidean Algorithm tells us to write p as an integral number of q's plus some remainder R_1—smaller than q. And then q is written as an integral number of R_1's plus some R_2 (smaller than R_1). Then R_1 in terms of R_2 plus some R_3 ($<R_2$) and so on, until the remainder R_n is zero. The remainder just before that, namely R_{n-1}, is the GCD of p and q. This method is best illustrated by an example:

Find the GCD of 58 and 5. We write:

$$58 = (11 \times 5) + 3 \tag{3}$$

$$5 = (1 \times 3) + 2 \tag{4}$$

$$3 = (1 \times 2) + 1 \tag{5}$$

$$2 = (2 \times 1) + 0 \tag{6}$$

So 1 is the GCD of 58 and 5, and the two numbers are relatively prime. To put these numbers into the equation required for the solution of this problem, we can proceed as follows:

From (5) we can write:

$$1 = (1 \times 3) - (1 \times 2) \tag{7}$$

From (4):

$$2 = (1 \times 5) - (1 \times 3) \tag{8}$$

Substituting into (7):

$$\begin{aligned} 1 &= (1 \times 3) - 1\,(1 \times 5 - 1 \times 3) \\ &= (2 \times 3) - (1 \times 5) \end{aligned} \tag{9}$$

And from (3):

$$3 = 58 - (11 \times 5) \tag{10}$$

Substituting into (9):

$$\begin{aligned} 1 &= 2(58 - 11 \times 5) - 1 \times 5 \\ &= (2 \times 58) - (23 \times 5) \end{aligned}$$

Q.E.I.
(*Quod erat inveniendum*)

8: Symmetry

Sam Loyd, who has been called America's Greatest Puzzlist, lived in the latter part of the nineteenth century and the early part of the present century. He was a good chess player in college, he later wrote a chess column for a New York newspaper. He discovered that he also had a flair for inventing puzzles and some of his puzzles are world-famous. The 14-15 puzzle is a rather well-known example. It consists of a square in which there are 15 smaller squares, which can be moved around, and one empty square. The 15 squares are numbered 1 through 15. The question is always to find whether or not the squares can be rearranged in a desired order. So popular was this puzzle that people neglected their regular duties to work on it. Shopkeepers neglected their shops, and in France, the Legislature was forced to ban the puzzle in order to get the legislators back to work. Sam Loyd's life and puzzles have been adequately described elsewhere. I include here Sam Loyd's Daisy Puzzle because it is not well known.

Sam Loyd's Daisy Puzzle

This is a game between two people and is played with a 13-petaled daisy. The players take turns picking the petals, and they may pick either any one petal or any two right next to each other, if such a pair exists. The

person who picks the last petal wins. What is the proper strategy for this game?

This game can be completely analyzed. The second player can always win, and the proper strategy thus becomes a recipe—a straightforward procedure which guarantees a win. The first player must take either one or two petals. The second player then takes two or one diametrically opposite. The first player then faces a daisy as is shown in figure 20, regardless of his first move.

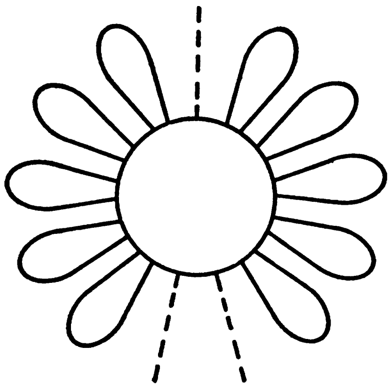


Figure 20

There are now two symmetrical groups of five petals each, and whatever the first player does in one group, the second player can do in the other. If the first player can take a petal, so can the second player, and therefore the second player always wins. It is quite apparent that the essence of his success is symmetry.

Cigars on Table

Another game for two people requires a supply of cigars and an ordinary table. The only restriction is that the table have two mutually perpendicular axes of symmetry. The rules of the game require each of the two

players to alternately place a cigar anywhere on the table, provided the cigar does not encroach upon any cigar already there. The first person who cannot legally place another cigar on the table loses the game. This is another game which is fairly simple to analyze, but in this case the first person can always win.

The first person places the center of his cigar at the center of the table, and the axis of the cigar is along one of the table's axes of symmetry. Each position on one side of this axis has a mirror image on the other side. Then, wherever the second player places his cigar, the first player puts his cigar in the mirror image of this position. So long as the second player can put down a cigar, the first player can put his in the mirror image. Thus, the first player is guaranteed victory.

Cup of Coffee and Cup of Tea

A problem which can generate violent arguments concerns the mixing of a cup of coffee and a cup of tea. We are given a cup of coffee and a cup of tea, containing equal volumes. We take a teaspoon of coffee, put it in the tea cup, and stir thoroughly. Then we take a teaspoonful of the mixture and put it back in the coffee cup. Does the coffee now have more tea in it than the tea has coffee, or vice versa?

An easy way to see that each cup of liquid is equally adulterated with liquid from the other cup is to forget about the details of the transfer, and consider only the final situation.

We started with equal volumes of liquid in each cup, and we ended with equal volumes in each. At the end, there is a certain amount of tea in the coffee cup. It must have replaced an equal volume of coffee, which was in the coffee cup at the beginning. And that missing coffee must now be in the tea cup.

Some people are not convinced by this argument. They point out that a full teaspoon of coffee was put in the tea, but then some of that coffee was brought back on the second transfer. And they say that there is then less than 1 teaspoonful of coffee in the tea cup. But they forget that because of this small amount of coffee coming back, there is not room for a full teaspoon of pure tea to mix in the coffee. A numerical example is very easy to work out and may be surprising.

Nim

Another problem game which uses symmetry is the game of Nim from the word "nim"—to filch. Nim is very old and well understood. The game is for two people and is played in the following manner:

Counters of some sort are arranged in N groups, each group having an arbitrary number. Each player, in turn, may take as many counters as he wishes *from any one group*. The player who takes the last counter wins (or loses, depending on what was decided at the beginning of the game). One of the more popular starting arrangements is shown in figure 21.

/ / / / /	101	
/ / / /	100	*Figure 21*
/ / /	11	
	212	

The rules for winning this game usually mystify the novice. The number of counters in each group is first written as a binary number, as is shown in figure 21. (Just as an ordinary, or decimal, number tells how many of each successive power of 10 to accumulate, a binary number tells how many of each successive power of two to accumulate.) Each column of binary numbers is then

added up in the decimal system. The sum will be one of two types. Each number in the sum will be even, or at least one digit will be odd. If each number is even, we will call it a "safe" situation, and if at least one number in the sum is odd, we will call it an "unsafe" situation.

The rules for winning require the aspirant to leave a safe situation to his opponent each time he moves. The opponent, facing a safe situation, must necessarily leave an unsafe one, because he must change at least one of the binary digits, and he can do so in only one group. The first player can then always change this unsafe total into a safe situation by taking counters from only one, carefully chosen, group. This can be seen by considering the digits in the sum of an unsafe group. There will be a left-most odd digit in the sum, and somewhere in the binary representations there must be a 1 in this column which corresponds to the odd digit. We choose to operate in any of the groups which has a 1 in this column. We can always remove a number of counters such that the offensive 1 is changed to a 0, and the binary digits to the right of that 0 can be whatever we choose. And we choose them so that the situation becomes safe. In this manner, the first player keeps control over the game until finally he faces a 2, 0 situation, or many groups of which only one has more than one counter in it. This is as far as the rules will take us, but it is far enough. The proper move in either of these situations can easily be chosen.

There is a way of playing Nim which does not require the binary notation and decimal addition to decide on the right move. This method is actually the same as the method which has just been described, but is somewhat easier for mental calculation.

Inspection of the groups reveals that group or those groups which have the highest power of two (a four or an eight, etc.) in them. Counters corresponding to this highest power of two must be arranged in pairs from

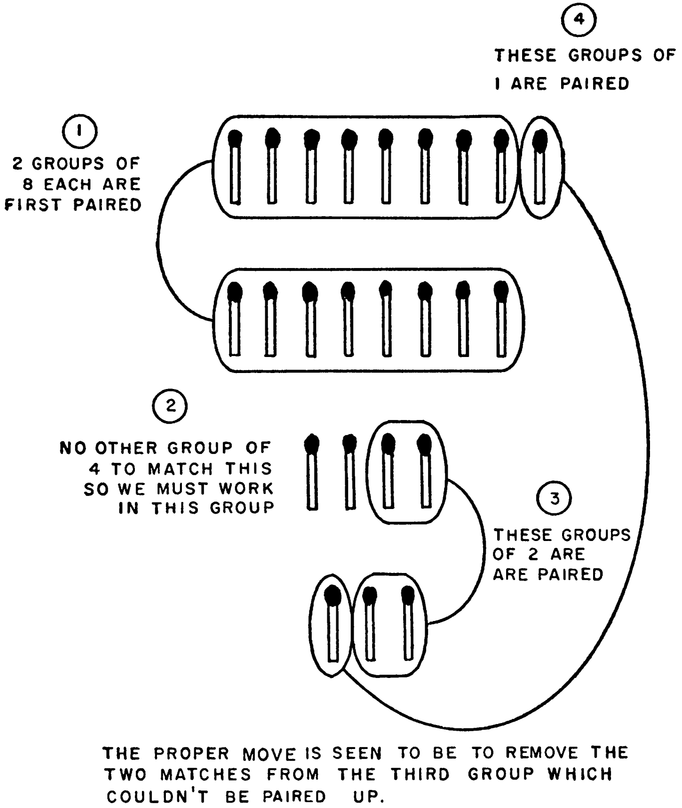


Figure 22a

different groups. When this highest power of two has been thus paired, the successively smaller powers of two must be similarly paired. (No power of two should be considered until all larger powers have been properly treated.) But the pairs of smaller powers of two are not required to be in the same groups as the larger powers of two.

Eventually, in an unsafe position, a power of two will be found which cannot be paired. Any of the groups which contain such a power of two can be chosen as the

group in which to operate. The remaining powers of two are paired, making no use, so far as is possible, of the chosen group. At the end of this process, the counters remaining in the chosen group are removed. This move will be the same as the one determined by the previously described method. Figure 22a shows an unsafe position, and the pairings that can be mentally chosen.

The binary representation for each of the groups of matches in the original position, and the decimal addition, are shown in figure 22b, and the representation for the safe

1001	1001
1000	1000
100	10
11	11
2112	2022

Figure 22b

position is also shown. The second method for winning at Nim emphasizes the use of symmetry in determining the proper move.

Nim$_k$

Strictly speaking, the game described above should be called Nim$_1$ because there exists a generalized version called Nim$_k$. Just as in Nim$_1$ we are allowed to take as many counters as we wish from any *one* group, so in Nim$_k$ we may take from as many as k groups if we wish. To win at Nim$_k$ we write the binary numbers as before, but now a safe position is defined as one in which each number in the decimal sum is divisible by $(k + 1)$. (Nim$_1$ is seen to be a special case of Nim$_k$.) The way to choose the group or groups to operate on is a generalization of the previously described method, and the limits at which this method must be modified are apparent.

The 21 Counters

Another game, similar to Nim, is the game in which a pile of 21 (or more or less) counters are supplied. Two people play the game and each person, at his turn, takes from the pile any number of counters less than a preassigned number, say 5. The object is either to take the last one or make the opponent take the last one. This game is very easily analyzed. The "key number" is 6. If the opponent takes c counters, then we can always take $6 - c$, regardless of what c is. In this way, control can be maintained. If we wish to take the last counter, then we must leave, for the opponent, some multiple of 6. If we wish the opponent to take the last counter, we must leave one more than a multiple of 6.

Grundy's Game

Grundy's Game is another game which is played with a pile of counters. Each person, in turn, must choose a group, and divide it into two unequal parts. The first move divides the entire pile of counters into two unequal parts. He who can no longer make a legal move loses.

If the game is played with a pile of 21 counters, the proper strategy to be followed in order to win at this game can be divided into two parts. First, we see that if we leave a pile containing 4, 7, or 10 counters to our opponent, then we can win. (When a single counter or a single pair of counters is left, it can be considered as "dead," because no further moves can change it. Thus, a group of 3 can be divided into 1 and 2, but then a further move is impossible.) Second, if two equal groups of counters are left to our opponent, then we can win, because whatever our opponent does in one group, we can do in the other.

A good strategy is to leave to the opponent a group of 4 or 7 or 10 counters, or any groups of these numbers, or any two equal groups which may be in combination with any or all of the above numbers.

Further investigation shows that (3, 6) counters, or (4, 7), or (5, 8) are also safe numbers of counters to leave to the opponent. (This can be shown by a process of exhaustion.) Since this is so, the first player can always win (we started with 21 counters). The first move should be to leave (9, 12). Then whatever the opponent does in any one group we can do in the other. In this way, the two numbers n and $n + 3$ can be reduced by taking equal quantities from each group, and this process must ultimately result in equal piles of counters, plus either (3, 6) or (4, 7) or (5, 8). In any of these cases, the game can easily be won. The analysis for Grundy's Game for an arbitrary starting position can be very complex indeed. It has been analyzed by T. H. O'Beirne in the British publication *The New Scientist,* and O'Beirne discusses this strategy in a book about to be published by the Dover Publishing Co. It does seem a pity that such a beautifully simple game has such a complex strategy.

Mutilated Chess Board

A fairly well-known, and almost standard, "war horse" type of problem is that of the "mutilated chess board," as Martin Gardner calls it. (It is not our intention to compile many standard problems which have been described elsewhere. But the new problem bears such a likeness to the mutilated chess board problem, that it is only fair and instructive to mention the previous problem.) This problem concerns a board divided into 64 squares. The two squares at the opposite ends of a main diagonal are cut out. Can the remaining area be completely covered

with 31 dominoes, each of which is two squares long and one square wide? The answer to this problem is "No." The proof is very beautiful, and almost devastating in its simplicity. Since the squares are alternately colored, each domino must cover a black and a white square. The number of covered white squares must be equal to the number of covered black squares. But the squares at opposite ends of a main diagonal are of the same color, so we are left with 32 squares of one color, and 30 of the other to cover. This cannot be done with 31 dominoes.

Construction of a Cube

The new problem involves the assembly of twenty-seven $1 \times 2 \times 4$ bricks into a $6 \times 6 \times 6$ cube. This problem was told to me by Gary Gordon of RCA-AE Division, and was originally invented by Professor R. Milburn of Tufts University. The question is simply: Can the 6-unit cube be formed with these 27 bricks?

$$(6 \times 6 \times 6 = 216 = 27 \times 8)$$

To see that such a construction is impossible (even though the combined volume of the 27 bricks is the same as that of the desired cube), we consider the $6 \times 6 \times 6$ cube to be colored as is shown in figure 23.

The individual cubes are colored in 27 groups of 8 each, and each group of eight is in the form of a small cube. (It may be convenient to consider the full cube to be subdivided into 27 $2 \times 2 \times 2$ cubes rather than one $6 \times 6 \times 6$, and for it to be colored as a three-dimensional checkerboard would be.) Each brick must necessarily occupy an equal number of the small black and white cubes. (This can best be seen by trying several different placements.) These bricks can surely not form portions of the $6 \times 6 \times 6$ cube, which does not contain equal numbers

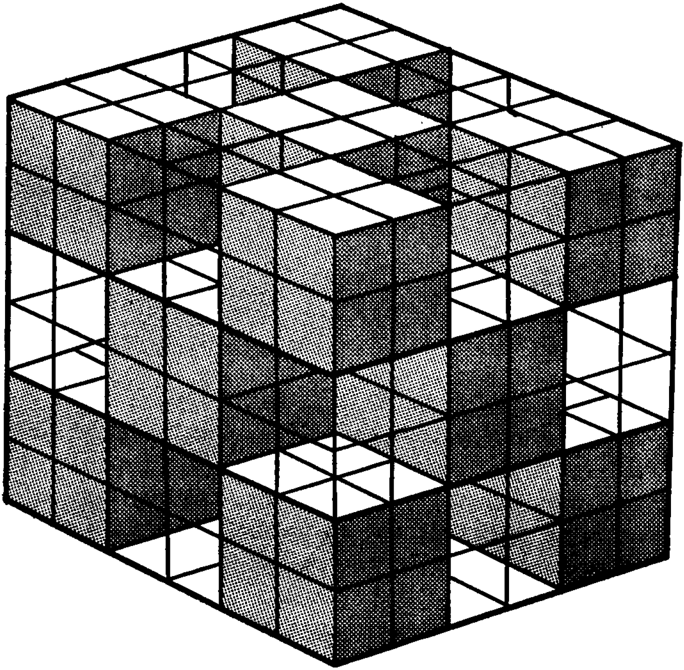


Figure 23

of small black and white cubes. But the full $6 \times 6 \times 6$ cube contains 14×8 cubes of one color, and 13×8 of the other. Hence, it cannot be formed of the 1 by 2 by 4 bricks.

Q.E.D.

9: Parlor Tricks and Number Manipulation

Eight 8's

Can you take eight 8's and arrange them in any fashion connected by the usual signs of mathematics so that they represent exactly one thousand? This question appeared recently in a popular magazine. The big question appears to be what, exactly, do we mean by "the usual signs of mathematics"? Do we mean only the plus, minus, times, and divide signs? Some people allow the use of the square root sign a finite number of times; others would allow its use an infinite number of times. We can consider the use of the factorial function and the gamma function. The implication in the problem, as I originally saw it, was that there was only one solution. This is very false. (If ever a superlative adjective needed to be compared, this is it.) The solution they were thinking of was the following:

$$888 + 88 + 8 + 8 + 8 = 1000$$

A much simpler solution would have been:

$$\frac{8888 - 888}{8} = M \ (1000)$$

Other solutions, which are only slightly more complex, are:

$$\frac{8 + 8}{8}(8 \times 8 \times 8 - 8) - 8 = M$$

$$8(8 \times 8 + 8 \times 8) - 8 - 8 - 8 = M$$

$$\left[(8+8)8 - \left(\frac{8+8+8}{8}\right)\right]8 = M$$

Still other solutions (which are equivalent to 10^3) are:

$$\left(\frac{88-8}{8}\right)^{(8+8+8)/8} = M \quad \text{and} \quad \left(8 + \frac{8+8}{8}\right)^{(8+8+8)/8} = M$$

If we use the factorial sign, we can get:

$$\frac{8!}{8\left(8 - \frac{8+8+8}{8}\right)} - 8 = M$$

$$\frac{8!}{8} - 8[(8 \times 8 \times 8) - 8] - 8 = M$$

$$8!\left[\frac{8+8}{8(88-8)}\right] - 8 = M$$

If we now allow ourselves to use the gamma function, we can get:

$$\Gamma(8) - (8 \times 8)\left(8 \cdot 8 - \frac{8}{8}\right) - 8 = M$$

$$\frac{8\Gamma(8)}{8+8+8+8+8} - 8 = M$$

We can use logs (base 8) and get:

$$[\log_8 8 \times 8][8 \times 8 \times 8 - 8] - 8 = M$$

$$[\log_{8\times8\times8\times8\times8} 8]\Gamma(8) - 8 = M$$

$$\left[\frac{(88-8)}{8}\right]^{[\log_8 (8\times8\times8)]} = M$$

$$\left[8 + \left(\frac{8+8}{8}\right)\right]^{[\log_8 (8\times8\times8)]} = M$$

And finally, we use the combinatorial symbol, where $\binom{n}{r} \equiv \frac{n!}{r!(n-r)!}$ and get:

$$\frac{1}{8}\left[\binom{8+8}{\frac{8}{.8}} - 8\right] + 8 - 8 = M$$

Possibly the most outrageous array of all is:

$$\left(\frac{88}{88} \times 8 \times 8 \times 8\right)_8 = 1000$$

Of course, this last isn't really legal because it represents one-zero-zero-zero in the number base 8. It looks like a thousand, but looks are often deceiving.

The Multiplication Theorem

The Multiplication Theorem of algebra, which is learned by all high school students (and forgotten by most), is very important for an understanding of many simple mental calculations. In one of its forms, it states that the product of two factors $(a + b)(c + d)$ is simply:

$$(a + b)(c + d) = ac + ad + bc + bd \qquad (1)$$

The case where $a = c$ and $b = d$ is a special case which deserves some note. It represents $(a + b)(a + b)$ or $(a + b)^2$. We can write:

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad (2)$$

The case where $a = c$ and $b = -d$ is another important and special case. This case reduces to:

$$(a + b)(a - b) = a^2 - b^2 \qquad (3)$$

Application of these three equations can be very useful in many examples of mental calculations.

As a demonstration of the usefulness of equation 3, the product of two numbers which differ by small, but equal, amounts from a number whose square we know, can quickly be stated. Thus:

$$19 \times 21 = (20 - 1)(20 + 1) = 400 - 1 = 399$$

It is hoped that most people know 20^2 by heart. This little trick does not seem to be of much interest in itself, but it becomes more useful when one realizes that not only are numbers ending in a zero easy to square, but numbers ending in a 5 are also easy to square. To square any such number, say 75, we merely multiply the number without the 5—(*i.e.*, 7 in this case) by the next higher number (8), and then affix 25 to the end of the resulting product. Thus:

$$7 \times 8 = 56 \qquad 75^2 = 5{,}625$$

An example which utilizes both equation 3 and this easy method of squaring numbers ending in 5 is the following We can write:

$$\begin{aligned} 73 \times 77 &= (75 - 2)(75 + 2) \\ &= 75^2 - 4 = 5{,}625 - 4 = 5{,}621 \end{aligned}$$

The generalization of this method of squaring numbers ending in a 5, to the multiplication of any two numbers ending in a 5, is reasonably simple and quite useful. In this latter case, we multiply together the two numbers without the 5's, and add to this product half the sum of the same two numbers. Then we affix 25 to the end as before. Thus:

$35 \times 55 = ?$

$$3 \times 5 = 15; \quad \frac{3+5}{2} = 4; \quad 15 + 4 = 19; \quad 35 \times 55 = 1{,}925$$

If the sum of the two numbers without the 5's happens to be an odd number, we take account of the $\frac{1}{2}$ in the sum

by affixing 75 to the end of the number rather than 25. For example:

$$35 \times 65 = ?$$

$$3 \times 6 = 18; \frac{3+6}{2} = 4\tfrac{1}{2};$$

$$18 + 4\tfrac{1}{2} = 22\tfrac{1}{2}; 35 \times 65 = 2{,}275$$

The proof of this method of multiplying numbers ending in 5 is simple.

Let us represent one of the numbers ending in 5 by $(10a + 5)$, and the other one by $(10b + 5)$. Using equation (1), the product of these two numbers is:

$$(10a + 5)(10b + 5) = 100ab + 50a + 50b + 25$$

$$= 100ab + \frac{100}{2}(a + b) + 25 = 100\left(ab + \frac{a+b}{2}\right) + 25$$

The particular case $a = b$, *i.e.*, the case of squaring a number ending in 5, gives the result:

$$100(a^2 + a) + 25 = 100a(a + 1) + 25$$

These results agree with the aforementioned rules.

These little tricks were particularly useful to me one day when a group of friends and I had worked out a problem in probability. The problem involved an "electronic decimal scaler," which consists of 5 columns of the digits from 0 to 9. The question was, if a perfectly random number appears in each of the 5 columns, what are the chances that it will represent nothing in poker, not a pair, not 3 of a kind, 4 of a kind, 5 of a kind. The 5 numbers could represent each of 5 cards in a game of poker with no suit designations. To calculate the probability of getting nothing we must recognize that the first number can be anything, the 2nd number cannot be the same, so it must be any of the remaining 9. The 3rd number must be one

of the remaining 8. The overall probability of getting nothing will be $(9 \times 8 \times 7 \times 6)/10^4$. This product, I stated immediately, was .3024. I told my stunned and impressed friends that I happened to notice that $7 \times 8 = 56$ and $6 \times 9 = 54$. The answer was obviously $10^{-4}(55^2 - 1) =$ .3024.

Some Useful Facts

Some other interesting facts concerning "smallish" numbers (*i.e.*, <50) and their squares, products, etc. are: If one knows the square of a particular number (n), then the square of the next higher is simply $2n + 1$ greater. In this way:

$$11^2 = 10^2 + 20 + 1 = 121$$

And the next higher square must be $4n + 4$ greater, *e.g.*:

$$12^2 = 10^2 + 40 + 4 = 144$$

Another occasionally useful fact is that the square of a number $(25 - n)$ differs from the square of $(25 + n)$ by an integral number of hundreds, equal to $100n$: *e.g.*, $24^2 = 576$, $26^2 = 676$, or $23^2 = 529$, $27^2 = 729$. This fact can easily be shown using the Multiplication Theorem:

$$(25 + n)^2 = 25^2 + 50n + n^2$$

$$(25 - n)^2 = 25^2 - 50n + n^2$$

$$\text{Difference} = 100n$$

In this manner, the squares from 26 to 49 can be simply gotten from a knowledge of the squares from 1 to 24.

A trivial but real example of how these methods can give startling results is the following true story. I once jokingly asked a fellow student, while we were studying in the same room, if he knew how much 2^{20} was.

"Sure," he said, "it's 1,048,576."

The fact that I was impressed was obvious. He got the answer so quickly by several rapid applications of the Multiplication Theorem. He knew, as do most mathematically inclined people, that $2^{10} = 1{,}024$. Hence,

$$\begin{aligned} 2^{20} &= 1{,}024^2 = (1{,}000 + 24)^2 \\ &= 1{,}000^2 + (2 \times 24 \times 1{,}000) + (24)^2 \\ &= 10^6 + (48 \times 10^3) + (24)^2 \end{aligned}$$

24^2 is easy to obtain by one of our previous methods. Hence:

$$2^{20} = 1{,}024^2 = 10^6 + (48 \times 10^3) + 576 = 1{,}048{,}576$$

Several years later, my friend confessed that, upon realizing his success in this case, he memorized the population of Alaska. He hoped to secure his reputation of being a wonderfully wise person, just in case I happened to wonder aloud what the population of Alaska was!

Many so-called "magic" tricks of a mathematical twist are easily proved with the aid of the Multiplication Theorem. One well-known trick, for example, requires a person to choose any three-digit number in which the first and third digits are not the same. He is then asked to reverse the three-digit number and find the difference between the reversed and original numbers. He is then requested to reverse the digits of the difference, treating it as a three-digit number if it is less than 100, and add the resulting number to the original difference. The unsuspecting accomplice now has the number 1,089, regardless of the original three-digit number. He could, for example, be requested to multiply the 1,089 by 40, because that might be the number of wives that a sultan, who has just consulted his prime minister concerning his insomnia, had. The mathematical fact, 1,089, can obviously be clothed in a great many stories. To prove that the final sum is 1,089, we can proceed as follows:

The original number, abc, can be represented as:

$$10^2a + 10b + c \quad or \quad 100a + 10b + c$$

Reversing this number yields:

$$10^2c + 10b + a \quad or \quad 100c + 10b + a$$

And the difference is:

$$100(a - c) + (c - a)$$

We can consider only $a > c$. Remembering that $(c - a)$ is then negative, we modify the expression for the difference so as to utilize only positive numbers.

We modify this by subtracting 100 and then adding 90 and 10. The difference can thus be written:

$$100(a - c) + (c - a)$$
$$= 100(a - c - 1) + 90 + (10 + c - a) \quad (1)$$

This difference can now be reversed:

$(10 + c - a)$ is the first digit

$(a - c - 1)$ is the last digit

The reversed difference is:

$$100(10 + c - a) + 90 + (a - c - 1) \quad (2)$$

The sum of (1) and (2) is:

$$100(\cancel{a} - \cancel{c} - 1 + 10 + \cancel{c} - \cancel{a}) + (2 \times 90)$$
$$+ (10 + \cancel{c} - \cancel{a} + \cancel{a} - \cancel{c} - 1)$$
$$= (9 \times 100) + 180 + 9 = 1{,}089$$

The multiplication theorem explains many simple so-called "magic tricks." It does not work, however, on a trick which I once played on my younger daughter. I asked her to think of a digit from 1 to 9. I then asked her to open her mouth, and after examining the inside of

her mouth carefully, I triumphantly announced that the number she had chosen was 6. In shocked disbelief she asked, "But Daddy, how did you do it?" She was so impressed that I didn't have the heart to tell her how lucky my guess was. Incidentally, there is a 10 percent chance of succeeding in this trick, and one's standing is so increased if he guesses correctly, that it seems worth while to try this scheme. (Keeping one's fingers crossed might help a little.)

Casting Out Nines

Another example which the Multiplication Theorem explains simply is the method of Casting Out Nines. Once familiar to all school children, this method is a simple check on the possible correctness of long multiplication problems. The sum of the digits in the multiplicand and the sum of those in the multiplier are first determined. If either of these sums has two or more digits, we take the sum again and again until there is a single digit left for each. This single digit is known as the digital root, or simply the *digital*, of the original number. The product of these two digitals must be equal to the digital of the product. If it is not, the multiplication must be wrong. If they are equal, the multiplication could be correct, but is not necessarily so. To illustrate the method:

$$\begin{array}{rll}
1234 & \text{(digital) } 1+2+3+4=10\rightarrow 1 & \\
\times 5678 & \text{(digital) } 5+6+7+8=26\rightarrow 8 & \\
\hline
9872 & & \\
8638 & \text{(product)} & 1\times 8=8 \\
7404 & & \\
6170 & & \\
\hline
7006652 & \text{(digital) } 7+0+0+6+6+5+2=26\rightarrow 8 &
\end{array}$$

Therefore this product 7,006,652 might be correct.

The proof of this method is straightforward. To start, we recognize that any number at all can surely be written as the sum of some integral number of nines plus some remainder. (It could also be written as some integral number of eights or sevens, plus some remainder, but let us consider only the nines.) We can thus write the multiplier N_1 and the multiplicand N_2 as:

$$N_1 = 9n_1 + R_1 \qquad\qquad N_2 = 9n_2 + R_2$$

The product N_1N_2 is, by equation 1:

$$N_1N_2 = 81n_1n_2 + 9n_1R_2 + 9n_2R_1 + R_1R_2$$

The most interesting feature of this expression is that each term in the product, except for the product R_1R_2, is divisible by nine. This is the same as saying that the remainder, after dividing the product by nine, must be the same as the product of the remainders of the original numbers when they were divided by nine. But the remainder of any number after dividing by nine is simply the digital of that number. Because of this easy way of getting the remainders after division by nine, this method of checking multiplication is quick and easy.

Other parlor tricks may require more or less ingenuity to explain than the illustrated examples do, but most can be explained in terms of the Multiplication Theorem.

10: Divisibility by Various Numbers

Divisibility by 2, 5, and 10

Divisibility of numbers by certain other numbers can often be checked mentally—very quickly. For example, everybody knows that if a number ends in a zero, it is divisible by 10. And likewise, if a number ends in a zero or a 5, it is divisible by 5. The reason is simply that any number multiplied by 10 ends in a zero, and any number multiplied by 5 ends in either zero or 5. If a number is divisible by 2, it must be even.

Divisibility by 3

Only if the sum of the digits comprising a number is divisible by three is the original number also divisible by three. The reason for this is most easily seen by considering what is really meant when we write a multidigit number. If the number is represented by *abcde* (where a, b, c, d, e are separate digits), we recognize that this is shorthand notation for:

$$a \cdot 10^4 + b \cdot 10^3 + c \cdot 10^2 + d \cdot 10^1 + e \cdot 10^0 = abcde$$

Any power of 10 has a remainder of one when it is divided by 3. Hence, any of the terms in the above expression, *e.g.*, $b \cdot 10^3$, will have as a remainder when divided by 3 its

coefficient (here b). Since the remainder after dividing each term by 3 is simply the coefficient, the remainder after dividing the whole number by three is simply the sum of the coefficients. If this sum is divisible by 3, then the original number must be divisible by 3.

Divisibility by 9

A method for checking and proving divisibility by 9 is also well known. Since any power of 10 has a remainder of one when divided by 9, it follows that the digital of a number must be divisible by 9 if the original number is divisible by 9.

Divisibility by 11

A rule for divisibility by 11 is not quite so well known as the two previous rules. In deriving this rule, we note that any even power of 10 (*e.g.*, 10^0, 10^2, or 10^4) has a remainder of one when divided by 11. And any odd power of 10 (*e.g.*, 10^1 or 10^3, etc.) has a remainder of 10 when divided by 11. Since a remainder of 10 can be considered to be a remainder of -1 (as far as divisibility by 11 is concerned), we conclude that the remainder left when dividing *abcde* by 11 will be:

$$a - b + c - d + e$$

Simply stated, the remainder is the difference between the sum of the digits in the odd places, starting from the right, diminished by the sum of the digits in the even places. For example, if 12,396 is divided by 11, the remainder is $(6 + 3 + 1) - (9 + 2) = 10 - 11 = -1$, or 10, which is the same thing. Only if the remainder is zero or divisible by 11 is the original number evenly divisible by 11.

Divisibility by 7

Rules for divisibility by other numbers can be derived, but they are unfortunately nowhere near as neat and simple as the rules for 10, 5, 2, 3, 9, and 11. As an example of one of these rules, let us consider the derivation of a rule for divisibility by 7. Let the original number be *abcdef*. As before, we write this out in full as:

$$(a \times 10^5) + (b \times 10^4) + (c \times 10^3) + (d \times 10^2) + (e \times 10^1) + (f \times 10^0)$$

The remainder after dividing 10^0 by 7 is 1, and so the remainder after dividing $(f \times 10^0)$ by 7 is $(f \times 1)$. In the same way, the remainder after dividing 10^1 by 7 is 3, and that after dividing $(e \times 10^1)$ by 7 is $3e$. In this way, we can calculate the remainders for each term in the above number, and the remainder of:

$$(a \times 10^5) + (b \times 10^4) + (c \times 10^3) + (d \times 10^2) + (e \times 10^1) + (f \times 10^0)$$

is:

$$(a \times 5) + (b \times 4) + (c \times 6) + (d \times 2) + (e \times 3) + (f \times 1)$$

(The remainders repeat periodically as we go to the left after $a5$, *i.e.*, the next number is multiplied by 1, the next one by 3, etc.) Thus, in the previous example of 12,396, the remainder after division by 7 will be:

$$(1 \times 4) + (2 \times 6) + (3 \times 2) + (9 \times 3) + (6 \times 1) = 55$$

And the remainder of this is 6 as can be seen in many ways. For instance, $(5 \times 3) + (5 \times 1) = 20$ and the remainder of 20 is $(2 \times 3) + (0 \times 1) = 6$. We can also see it by simply noting that $55 = (7 \times 7) + 6$.

The above formula can be made slightly (but only slightly) more presentable by replacing some large positive remainders by smaller negative ones. Since a remainder

of 6 is the same as a remainder of -1 (as far as divisibility by 7 is concerned) and a remainder of 5 is the same as -2, and 4 the same as -3, we can rewrite the remainder as: $-2a - 3b - c + 2d + 3e + f =$ the remainder of *abcdef* after division by 7. As before, the remainder of 12,396 is:

$$-(3 \times 1) - (1 \times 2) + (2 \times 3) + (3 \times 9) + (1 \times 6) = 34,$$

and the remainder of this is $(3 \times 3) + (4 \times 1) = 13$ or 6. By the simplified formula, as with the original formula, the remainder after division of 12,396 by 7 is 6. The test for divisibility by 7 can be put into somewhat more concise form, by the following scheme: Divide the total number into groups of three digits each, starting from the right. In each one of these groups, multiply the right hand number by 1, the middle number by 3, and the left hand number by 2. Then, the sum of the alternate triads diminished by this sum of the intermediate triads is the remainder. For example, in the number 12,396, we first write:

$$\underbrace{0\ 1\ 2}\ \underbrace{3\ 9\ 6}$$

We then multiply by 2, 3, and 1 to get:

$$\underbrace{0\ 3\ 2}\ \underbrace{6\ 27\ 6}$$

The alternate sum in this case is simply the difference of $6 + 27 + 6$ and $0 + 3 + 2 \rightarrow 39 - 5 = 34$.

The remainder of 34 is $\underbrace{0\ 9\ 4} \rightarrow 13 \rightarrow \underbrace{0\ 3\ 3} \rightarrow 6$

Most people will agree that this is not a cute and elegant method. But it works.

Any number of the form *abcabc*, where *a*, *b*, and *c* are any integers from 0 to 9, is divisible by 13. This statement is sure to be greeted with skepticism by some people. Not only, however, is it divisible by 13, but it is also divisible by 7 and 11. It is most convenient to write out the complete expression for which *abcabc* is the abbreviation. The complete expression is:

$$abcabc = a \cdot 10^5 + b \cdot 10^4 + c \cdot 10^3 + a \cdot 10^2 + b \cdot 10^1 + c \cdot 10^0$$

In this expression, we can factor out $(10^3 + 1)$ and we finally arrive at:

$$abcabc = 1001(a \cdot 10^2 + b \cdot 10^1 + c \cdot 10^0)$$

The number 1001 has the prime factors 7, 11, and 13. Since 1001 is a factor of the number, so are 7 and 11 and 13 and any multiplicative combinations of these numbers. This fact can be used in a great many different ways to form interesting problems. We could, for example, ask a person to write down a 3-digit number, and then to repeat those same three digits to form a 6-digit number. Then, we look at it and we can say, "Aha! This number is divisible by 143." (We know that 143 is the product of 11 and 13, but we don't have to tell it to the victim.)

We could also look at the number and say, "Oh, that's divisible by 77." The fact that 7 and 11 are known factors and that the product 77 is, therefore, also a factor doesn't have to be stressed. This idea is used by Martin Gardner in his "Mathematical Games" column in *Scientific American* magazine to describe an amusing parlor trick.

The "wizard" asks someone to write any 3-digit number on a piece of paper. Then, a second person is asked to repeat the same 3 digits to form a 6-digit number, and to divide that 6-digit number by 7. [*We* know that the remainder will be 0, but the mystery may be introduced by noting the remainder (0) on a piece of paper and giving it beforehand to another person to hold.] A third person is asked to divide the second person's answer by 13, and to note the remainder of that operation. And then, the first person is asked to divide that second quotient by 11 and to note that remainder in the same place. If this first person reacts as most people do, he will be shocked to find that the quotient he obtained is the 3-digit number he first wrote down. The other two people will be surprised to a

somewhat lesser degree when they discover that the result of both of their divisions had a zero remainder.

Another similar problem is the following: Prove that if we take any number and rearrange its digits to form any other number, the difference between the original number and the second number is always divisible by 9. We shall consider the first arbitrary number to be represented by *abcdefg*. The rearranged number might be *fadbecg*. The difference between these two numbers must necessarily be a sum of terms, each of which consists of one of the original digits times the difference between two powers of 10, *e.g.*, $a(10^6 - 10^5)$ or $b(10^5 - 10^3)$. The difference between two powers of 10 is always either zero or a string of nines followed by a string of zeros. Either case is divisible by 9 and so the whole number must be divisible by 9.

As our last example, we can consider the following problem:

Examine the number:

$$8,-5-,383,-2-,936,-5-,8-2,03-,9-3,-76$$

Each of the ten blank spaces in the above number is to be filled in with a different one of the digits from zero to nine. What is the probability that the resulting number will be divisible by 396?

The first reaction to this problem is that it represents the height of idiocy or the acme of stupidity, whichever you prefer. Actually, however, the problem solver has the advantage in that we suspect that no one in his right mind would even propose such a problem unless it had a reasonably clever solution. This is actually the case. First, we notice that the number ends in a 76. So the total number must be divisible by 4. Secondly, the sum of the digits from zero to nine is known to be 45, and the sum of the digits that are given is equal to 90, so that the sum of all the digits is 90 plus 45 = 135, regardless of the order in

which the digits from zero to nine are inserted. Since 135 is divisible by 9, the original number must be divisible by 9. It will be noticed that the blank spaces occur with Machiavellian regularity only in the second, fourth, sixth places, etc. — always in the even places. We can thus determine that the sum of the digits in the odd places is 73, and the sum of the digits in the even places is 17 plus the sum from zero to nine, or 17 plus 45 = 62. Since the difference between 62 and 73 is 11, the original number must be divisible by 11, regardless of the order in which the digits from zero to nine were inserted. The known factors are thus 4, 9, and 11, and the number is therefore divisible by the product of these numbers, 396. The probability is thus seen to be unity.

11: Problems Which Seem to Have No Solution

The Careless Indian Who Drops His Paddle

An Indian, canoeing upstream, accidentally drops one of his paddles. He does not realize this until ten minutes later. At that time, he turns around and paddles downstream (at the same rate relative to the stream), and catches up with the paddle one mile from the place he dropped it. What is the rate of the stream?

One of the easiest ways to solve this puzzle is to consider the flowing stream to be a conveyor belt, and the Indian to be walking on it. On the belt, he drops the paddle, walks away for ten minutes, and then returns. His total trip has taken him twenty minutes, and the paddle (or the conveyor belt) has moved one mile in this time. The rate of the stream is one mile in twenty minutes or three miles per hour.

The Early Commuter

A man commutes to work, and each day his chauffeur leaves the house in time to meet his master at the station at exactly 5 o'clock. The chauffeur then turns around (instantaneously, of course) and takes the man home. One day, the man comes into the station early, and starts to

walk home. On the way, he meets the chauffeur, who had left home at the usual time. He gets in the car and arrives home ten minutes earlier than usual. At what time did he meet the chauffeur?

The chauffeur drives at the same speed going to and from the station. If the chauffeur got home 10 minutes early, then he must have saved five minutes going to and coming from the station. Since he normally arrived at the station at 5 o'clock, he met his master and turned around at five minutes before 5.

This solution can be represented graphically as is shown in figure 24. In this figure, the path AB represents the chauffeur's normal trip to the station, and BC the trip back. The time at B is exactly 5 o'clock. On the day in question, the chauffeur travels AED. DC is given as ten minutes. Since ABC and AED are both isosceles, E must occur at five minutes before 5 o'clock.

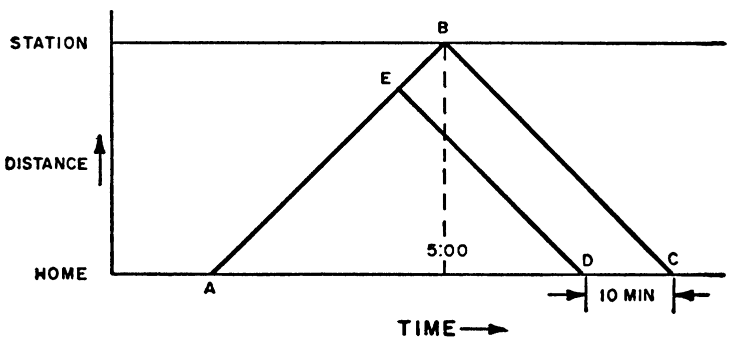


Figure 24

The Officer and His Column of Men

A column of soldiers is one mile long. The officer in charge, standing at the rear of the column, gives the order to advance. He starts to march in the same direction, but at a faster speed. When he reaches the head of the column,

he turns around and marches toward the rear. When he reaches the tail end of the column, he notices that the tail is now where the head was when they started. How far has the officer marched?

The diagram in figure 25 shows the officer and the column of troops at three different times.

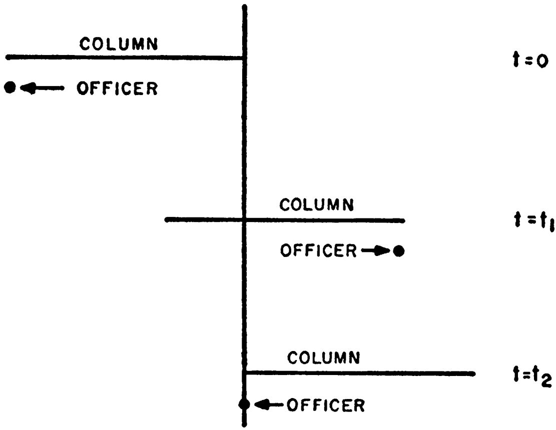


Figure 25

Let us denote the velocity of the column by v_c and the velocity of the officer by v_o. The rate of the officer with respect to the column when they are going in the same direction is $(v_o - v_c)$. So t_1 must be the length of the column (one mile) divided by this velocity, *i.e.*, $1/(v_o - v_c)$. The rate of the officer relative to the column when they go in opposite directions is $(v_o + v_c)$. The time interval $(t_2 - t_1)$ is $1/(v_o + v_c)$. The total time from zero to t_2 is then:

$$t_2 = \frac{1}{v_o - v_c} + \frac{1}{v_o + v_c}$$

Time from 0 to t_2 is also equal to $1/v_c$ because the column advanced one mile. We can then write:

$$\frac{1}{v_c} = \frac{1}{v_o - v_c} + \frac{1}{v_o + v_c}$$

Let us multiply by v_o and let $d = v_o/v_c$:

$$d = \frac{v_o}{v_o - v_c} + \frac{v_o}{v_o + v_c}$$

Dividing numerator and denominator by v_c we get:

$$d = \frac{d}{d-1} + \frac{d}{d+1}$$

We recognize that $d = [v_o \times (1/v_c)]$ is the distance that the officer walked:

$$d = (d^2 + d + d^2 - d)/(d^2 - 1).$$

Simplifying, we get $d^2 - 2d - 1 = 0$

$$d = (2 \pm \sqrt{8})/2 = 1 \pm \sqrt{2}$$

The officer walked $1 + \sqrt{2}$ miles.

The Master Painter Vs. the Apprentice*

A master painter can put two coats of paint on a particular barn in one day. He normally paints one coat before lunch and one coat after lunch. Each coat takes the same length of time to apply. One day he hires an apprentice two hours before lunch. He fires him after the first coat is entirely applied. He then goes on to finish the barn himself, observing the usual lunch time (he uses fast drying paint, so that he can start the second coat immediately). One hour before his usual quitting time, he notices that he has applied the second coat everywhere except where the apprentice had painted in the morning. The apprentice

* This problem, incidently, was invented by V. D. Landon to fill an apparent void in such problems coming to our attention.

paints 200 square feet per day. What is the area of this barn?

The area that the apprentice painted in the morning must have been as much as the master painter could have painted twice (two coats) in one hour. The master painter would then have taken one half hour to paint this area once. The master painter would have finished one coat on the entire barn himself by lunch time. So he would have painted everything except what the apprentice did by half an hour before lunch. (The apprentice must, therefore, have been fired half an hour before lunch.) In other words, the apprentice took $1\frac{1}{2}$ hours to paint what the master painter could have done in $\frac{1}{2}$ hour. Since the apprentice paints 200 square feet per day, the boss can paint 600 square feet per day; the area of the barn must be 300 square feet.

The Two Ferry Boats

There are two ferry boats, F_1 and F_2, on opposite sides of a river. They travel at different rates, v_1 and v_2. They start to cross the river at the same time and pass each other 720 yards from one side. Upon reaching the opposite

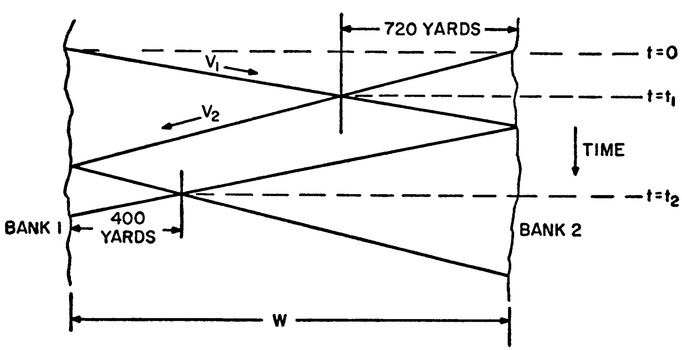


Figure 26

banks (at different times, of course) they turn around and go toward the opposite shore. This time they pass each other 400 yards from the other bank of the river. How wide is the river?

The diagram on figure 26 shows how equations can be simply derived.

Since the boats start at the same time, we can write:

$$v_2 t_1 = 720 \qquad v_1 t_1 = w - 720$$

or:

$$\frac{v_2}{v_1} = \frac{720}{w - 720} \tag{1}$$

We can also write for the second intersection at t_2:

$$v_2 t_2 = w + 400 \qquad v_1 t_2 = 2w - 400$$

or:

$$\frac{v_2}{v_1} = \frac{w + 400}{2w - 400} \tag{2}$$

Equating (1) and (2):

$$\frac{720}{w - 720} = \frac{w + 400}{2w - 400}$$

$$1440w - (720 \times 400) = (w + 400)(w - 720)$$

$$1440w - (720 \times 400) = w^2 - 320w - (720 \times 400)$$

The large constant terms cancel, and we are left with:

$$w = 1760 \text{ yards}$$

Volume of a Modified Bowling Ball

A standard size bowling ball has a hole drilled straight through the center. The side of the finished hole is three inches long. What is the volume of the bowling ball after the hole has been drilled out? Protestations saying, "You

didn't give me the original diameter of the ball!" or, "You didn't tell me the diameter of the hole (or the drill)" are common. But these numbers are not required for a solution.

A straightforward calculation, using calculus, of the volume of the ball remaining would show that this volume depends only on the length of the side of the hole and not on the diameter of the hole or the original diameter of the ball. Another argument can be advanced by those who prefer not to use the calculus. One can say, "This chap doesn't give problems that have no solution. Since he didn't give me the diameter of the hole, the volume of ball remaining must be the same, no matter how big the diameter is (as long as the side is three inches long). If this is so, then I can assume that the hole is an exceedingly thin hole (perhaps vanishingly small). And then the volume of the ball remaining would be simply the volume of a sphere whose diameter (the length of the vanishingly small hole) was exactly three inches, and this volume is:

$$\frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left(\frac{3}{2}\right)^3 = \frac{9\pi}{2}$$

12: Probability Problems

Die Throwing

How many times, on the average, must one toss a six-sided die before every number from one to six comes up at least once?

The solution to this problem is quite straightforward, but perhaps it can be made even easier to accept if we consider a simpler version of the same problem first. Namely, how many times, on the average, must a coin be flipped before at least one head and one tail appear? We note that the first throw must be either a head or a tail, and we wish to know how many additional tosses are required, on the average, before we get the other one. The probability of getting the other one is $\frac{1}{2}$. If S represents the number of successes and N the number of trials, probability can be defined as the limiting value of the ratio S/N, as N tends toward infinity. Or probability is $\overline{S}/N$ where $\overline{S}$ is the average number of successes in N trials. In general, $p = \overline{S}/N$. In this problem, we want to know the average number of tosses to get one success if the probability is $\frac{1}{2}$. Since $N = \overline{S}/p$, this number is $1/\frac{1}{2} = 2$. Thus, the answer to the simpler version is one (for the first toss) plus two (to get the other one) equals three.

Returning to our original problem; on the first throw, we'll get one of the six numbers, we don't care which one. Having that one, the probability of getting a number

different from the first one is $\frac{5}{6}$. On the average, then, $\frac{6}{5}$ of a throw are required to get something different from the first. After this second number, the probability of getting something different from the first two is $\frac{4}{6}$, and we expect to require $\frac{6}{4}$ of a throw to get it. In this way, the total number of throws will be 1 (for the first) plus $\frac{6}{5}$ (for the second) plus $\frac{6}{4}$, etc., or:

$$T \text{ (number of throws)} = 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 14.7$$

There are bound to be skeptics who do not believe this solution. An experimental test of the coin tossing problem is much easier than a test of the die throwing problem.

Bertrand's Box Paradox

In "Bertrand's Box Paradox" there are three boxes, all of which look alike on the outside. One is known to contain two gold coins (*gg*), the second, a gold and a silver (*gs*), and the third, two silver coins (*ss*). One of these boxes is chosen at random, and a single coin is withdrawn. That single coin turns out to be silver. What is the probability that the coin remaining in that box is also silver?

Most people reason in the following manner: If the first coin turns out to be silver, it must have been taken either from the *ss* box or the *sg* box. Since the probability of choosing any one box is the same as the probability of choosing any other, and we know that one of only two boxes must have been chosen, the probability that the remaining coin is silver must be one half. This reasoning is, of course, incorrect. (If the incorrect solution were not so attractive, I would not have included such a simple example.)

The reasoning above is incorrect (even though many famous mathematicians were fooled) because it is *not* equally likely that the *ss* or the *sg* box has been chosen (if

we know that a silver coin has been chosen). It is, in fact, twice as likely that the *ss* box has been chosen rather than the *sg*. An easy way to see this is to consider the procedure of picking a box and choosing a coin to be repeated a large number of times—say, six thousand. Half of all these times, a silver coin will be chosen, and each box will be chosen a third of all the times. But in only half of the number of times in which the *sg* box was chosen will a silver coin be withdrawn. So, we have two thousand cases where the *ss* box was picked (and the silver coin withdrawn), but only half of two thousand for the *sg* box. It is clear from this analysis that in two thirds of the cases the *ss* box has been picked.

Another method of reasoning: If the original statement had neglected to tell what kind of coin had been picked, and had simply asked for the probability that the remaining coin is the same as the first one, the answer would clearly be two thirds, because we are interested in either the *ss* or the *gg* box. If a golden coin were withdrawn, the probability would be two thirds that the remaining coin was gold. And similarly, if a silver coin were withdrawn, the probability would be two thirds that the remaining coin was silver. Since it makes no difference what kind of coin was chosen, the statement that "the coin was silver" contains no additional information and cannot change the probability. The probability is thus two thirds.

Probability of Identical Pairs Among Cards

Suppose we have two shuffled decks of cards face down, and side by side, and we compare the first, second, third, etc. cards of each deck. What is the probability that at least one pair of these cards will be an identical pair (*e.g.*, queen of spades and queen of spades)? The

solution to this problem can be very messy and difficult if we do not use the right approach. For the sake of simplicity, we can consider the cards to be numbered from 1 to 52, and only one deck need be shuffled—the other can be in its natural order. The probability that the first card bears the number 1 is $\frac{1}{52}$ (because there are 52 numbers possible, and only one is correct). The probability that the first card does *not* bear the number 1 is $[1 - \frac{1}{52}]$. The probability that each succeeding card does not correspond to its order in the deck is nearly $[1 - \frac{1}{52}]$. (It is only "nearly $[1 - \frac{1}{52}]$" because we have neglected the possibility that the number r has appeared *before* the r^{th} card was inspected. The probability would be unity (1) then, that the r^{th} card does not bear the number r. But the assumption that the probability is $[1 - \frac{1}{52}]$ so simplifies the calculation of the total probability, that we yield to the temptation.)

The probability that none of the 52 cards bears a number corresponding to its order is the product of all these probabilities, or $[1 - \frac{1}{52}]^{52}$.

The evaluation of this quantity is somewhat simpler than it appears offhand. The expression $[1 + (1/n)]^n$ has an unexpected behavior. It can be evaluated for $n = 1$, 2, 3, etc., and it is found that the value, instead of increasing beyond all bounds, approaches 2.71828 . . .

This number is the celebrated number known as e. If we consider the fraction $[1 + (x/n)]^n$ then this function chokes off at a value of e^x. In the original expression that we had to evaluate, namely, $[1 - \frac{1}{52}]^{52}$, it can be shown that the value is very close to $e^{-1} = 1/e$. Since e is equal to 2.71828, the probability that *no* identical pairs will be found is somewhat greater than one third, and the probability that at least one identical pair *will* occur is somewhat less than two thirds. This rarely fails to shock people unfamiliar with this problem.

Probability of Same Birthday Among N People

How large a group of people must be considered in order to have a 50-50 chance that some pair in the group has the same birthday? In this problem, as in the last, it is worth while to consider the probability that the event will *not* occur. The first person can have any birthday during the year. The probability that the second person does *not* have the same birthday is $\frac{364}{365}$. And the probability that the third does not have the same birthday as *either of the first two* is $\frac{363}{365}$. The probability that no pair among a group of N people has the same birthday is:

$$P = 1 \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \ldots \times \frac{365 - (N - 1)}{365}$$

If we set P equal to one half and solve for N, N turns out to be nearly 23. So the chances are a little better than 50-50 that some pair of people in a group of 23 have the same birthday. (Don't forget that any two of the group might make up the pair, it might be the 6th and the 20th.)

St. Petersburg Paradox

The St. Petersburg Paradox is a game in which the player tosses a coin, and keeps tossing so long as it comes up heads. The game is over when he tosses his first tail. The banker pays the player 2^N dollars, where N is the number of heads he tossed before the first tail. If the player gets a tail on the first shot, he gets $2^0 = 1$ dollar. The question is: What is a fair price for the player to pay the banker for each game?

To answer this question, we make use of a concept known as the mathematical expectation. The expectation is the sum of the products of the probability of each pos-

sible outcome times the gain for that event. In this case, the probability of getting N successive heads and then a tail is $1/2^{N+1}$, and the gain is 2^N dollars.

$$\text{Expectation} = \sum \text{probability} \times \text{gain}$$

Thus, the expectation is

$$\sum_{N=0}^{\infty} \frac{2^N}{2^{N+1}} = \sum_{N=0}^{\infty} \frac{1}{2^1} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} + \ldots$$

and this expression clearly diverges. It is larger than any finite number we can choose. The paradox is that the expectation should be the fair price to pay, but no one but a fool would pay an infinite amount of money for the privilege of playing this game. What has gone wrong? Is our calculation incorrect? In spite of the fact that the calculation is so simple, it does contain an error. It tacitly assumes that if the player were to get 50 heads in a row, the banker would be able to pay him 2^{50} dollars. But this is more money than exists in the world! Our calculation must take into account the fact that there is a maximum of money which the banker can pay. If we assume this maximum is \$1024 (for ten heads in a row) then the calculation of the mathematical expectation must be modified in the following manner: The mathematical summation must be broken into two parts. The first is calculated in the usual way up until $N = 10$, and then we must add on to this a summation of $1/2^{N+1}$ times a fixed gain of 2^{10}. This can be written:

Expectation

$$= \left[\sum_{N=0}^{10} \frac{1}{2^{N+1}} \times 2^N\right] + \sum_{N=11}^{\infty} \frac{1}{2^{N+1}} \times 2^{10} = 5\frac{1}{2} + \frac{1}{2} = 6$$

and this says that the fair price for each game would be six dollars. (This is quite a bit more reasonable than an

infinite sum of money, but it still sounds rather high.) Another point we should recognize about this game is that our expected earnings depend upon the number of times the game is played. On the average, a string of five heads and a tail would be expected only once in every 2^6 games, and we would hardly be justified in expecting such a string in four or five games. We can also calculate the average number of throws per game. The expression for the average number of throws per game is the sum of the probabilities of the game lasting N tosses times N. So, on the average, the number of throws would be equal to:

$$\left(\frac{1}{2} \times 1\right) + \left(\frac{1}{4} \times 2\right) + \left(\frac{1}{8} \times 3\right) + \left(\frac{1}{16} \times 4\right) + \dots + \left(\frac{1}{2^N} \times N\right) + \dots$$

Count Buffon was curious about the mathematical expectation in the St. Petersburg game. He arranged for a child to play the game 2,048 times, and found that the total payment (of bank to player) was 10,057 units. The expectation is thus approximately 5. This is in fair agreement with the ideas expressed in this section.

EVALUATION OF $\sum_1^{\infty} \frac{N}{2^N}$

This expression can be evaluated by writing the sum as shown on the facing page.

The vertical sum of the terms on the left is seen to be equal to the series we wish to evaluate, and the sum on the right, which is the average number of throws per game, is equal to 2.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots+\frac{1}{2^N}+\ldots=1$$

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots+\frac{1}{2^N}+\ldots=\frac{1}{2}$$

$$\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots+\frac{1}{2^N}+\ldots=\frac{1}{4}$$

$$\frac{1}{16}+\frac{1}{32}+\ldots+\frac{1}{2^N}+\ldots=\frac{1}{8}$$

$$\frac{1}{32}+\ldots+\frac{1}{2^N}+\ldots=\frac{1}{16}$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\ldots+\frac{N}{2^N}+\ldots=2$$

* (Read It to Find Out)

The number * has been associated with tragedy since time immemorial. For this reason, the *th floor of a hotel is often omitted. The actual damage brought on by lack of respect for this number may be long delayed and very long lasting—as, for example, in the case of the "Sleeping Beauty." In this case, you remember, the wicked old witch (who was the *th fairy, and whom the regents didn't invite to their child's christening) cast a spell on the child which didn't take effect until 16 years later, and which lasted for one hundred years. (They don't make spells like that any more.)

In view of the dangers associated with *, it would seem prudent for authors to avoid such a chapter in their books. But people have been blithely ignoring these dangers, and have been writing *th chapters with no regard for the awesome events of the past, and with no apparent ill effects to themselves. It therefore seems probable that the terrors exist mainly in folklore and superst|tion, and not in reality.

But what's the point in taking chances?

* is intended to denote the decimal equivalent of the binary number 1101.

14: Humor in Mathematics

Liars and Truth-Tellers

The problem of the liars and the truth-tellers is a fairly old one. A missionary is shipwrecked on an island where there are two tribes of natives—the liars and the truth-tellers. The liars always lie when they speak, and the other tribe always tells the truth. The missionary wishes to go to the town of "A," but as he goes, he comes to a fork in the road. He sees a native standing there, but doesn't know to which tribe the native belongs. What one question can the missionary ask to find out which road to take, regardless of which tribe the native belongs to?* The answer is that the missionary asks, "Which road would another member of your tribe tell me to take if I wanted to get to the town of 'A'?"

If the native were a truth-teller, he would tell the truth. And, if he were a liar, he would lie about what a liar would say, and thus he, too, would tell the truth. The missionary finds out which road to take, but he doesn't know whether the native was a liar or not.

* If you take me to task for ending this sentence in a preposition, I will counter with a story about the small boy who was sick and in bed. His mother brought a book from which she could read to him. He didn't like the book, and he petulantly asked, "Mother, what did you bring that book I didn't want to be read to out of up for?"

Liar or Truth-Teller—at N-tuple Intersection

The same missionary on the same island comes to an n-tuple intersection of roads. He sees the same native (of unknown honesty). What *one* question can he ask the native to find which road to take and whether the native is a truth-teller or not?

The missionary asks, "Which of all these roads does *not* lead to the town of 'A'?" The liar will point to the one road that does lead to "A," and the truth-teller will point to all the other roads except the one that *does* go there. In either case, the single road to take is clearly indicated, and the multiplicity of the answer is inversely proportional to the honesty of the native! One might think that Information Theory would forbid getting so much information from a single question. But the fact that the question does not have a simple yes or no answer makes the analysis more complex than first meets the eye.

Where on Earth Was He?

A hunter travels 10 miles south, 10 miles east, 10 miles north, and shoots a bear. He discovers that he is back exactly where he started. What color was the bear? The traditional answer is that the bear was white, because the only place where one can walk as described and come back to the starting point at the end is the North Pole. This answer will undoubtedly satisfy many people, but the question we ask is: how many such starting points are there on the earth?

Consider a circle of latitude near the South Pole, exactly 10 miles long. Then, all points on the circle of latitude 10 miles north of this circle will be points that fulfill the conditions of the problem. The hunter goes

south, around the 10-mile circle of latitude (ending where he began) and then north to the point from which he started. The upper circle of latitude, of course, contains an infinite number of points. And if this infinity of answers were not sufficiently deflating, we can consider those circles of latitude 10 miles north of circles whose circumference is 10/2, 10/3, etc., miles. (But at any of these latter points, his "bear" would have looked more like a penguin.)

Goldbach's Conjecture

A German physician named Goldbach noticed that in every case he tried, an even number could be represented as the sum of two, and only two, prime numbers in at least one way. Thus:

$$20 = 7 + 13, \quad 22 = 11 + 11, \quad 24 = 7 + 17, \quad \text{etc.}$$

Goldbach wrote to the mathematician Euler mentioning this curiosity, and asked if Euler could prove it. Euler is said to have written back seventeen years later, saying, "No." Although many famous mathematicians have tried to prove the "Goldbach Conjecture," none of them has had any success at all, until very recently.

The first partial success was reported by a Russian mathematician named Schnirelmann, who proved, in 1931, that any even number could be written as the sum of not more than 300,000 primes! And to top this, another Russian named Vinogradoff proved, in 1937, that *nearly* every even number could be written as the sum of no more than four primes. The meaning of the word "nearly" in a mathematical proof could be that the fraction of all even numbers less than N which could *not* be represented as the sum of four primes approaches zero as N approaches infinity. Work on the Goldbach conjecture is still going on, but there do not seem to be any startling new developments.

A "Seemingly Impossible" Problem

An article in the *American Mathematical Monthly** recently reviewed the methods used in Russia to teach mathematics. At the end of the article there was a list of problems typical of those given in Russian math finals. One problem gave the sequence 100001, 10000100001, 1000010000100001, etc., and asked for a proof that all numbers of this sequence are composite (*i.e.*, not prime).

We can represent the numbers in this sequence by G_n where n is the number of groups of four zeros in the number. It is easily shown that G_1 is divisible by 11, G_2 by 111, G_3 by 1,111. But G_4 is not divisible by 11,111. Somewhat more sophisticated methods can be used to show that G_5 and all higher G's are composite. But G_4 is adamant in refusing to yield to a proof that it, too, is composite. Many people tried their luck with this problem and had a signal lack of success. But one day, John Waite, an engineer at RCA, showed me two ridiculous numbers, and with a grin broader than the one on the infamous cat, asked me to multiply them. The numbers were:

$$4{,}672{,}725{,}574{,}038{,}601 \text{ and } 21{,}401$$

To say that I was shocked to find the product equal to G_4 ranks as an understatement comparable to General Custer's reputed "Over that hill, fellows, I think there are friendly Indians!"

The original problem may have been a "ringer," and may have been included merely to see how far the youngsters could get. Or perhaps Russian educators didn't realize how tough the problem really is. The method used by John Waite in cracking this painfully large number is still a mystery to me, but I know it leans heavily on studies by Gauss and by Euler of the composition of very large numbers.

* *American Mathematical Monthly, 64* (1957), p. 397, Problem 8.

Archimedes' Cattle Problem

"Archimedes' Cattle Problem" considers a herd of eight differently colored bulls and cows, and gives seven simple arithmetical relationships between them (*e.g.*, the number of white bulls equals $[\frac{1}{2} + \frac{1}{3}]$ the number of black bulls plus the number of yellow bulls). It then asks for the number of each of the differently colored oxen. The solution to this part of the problem is laborious, but within reach, each number being of the order of several million. The original problem, however, puts additional restrictions on the numbers by saying the number of white bulls plus the number of black bulls is a square number, and the number of dappled bulls plus yellow bulls is a triangular number. (A triangular number is one of the set 1, 3, 6, 10, 15 . . . $\sum_{1}^{N} n$, so called because they fit nicely into a triangle, as for example, do ten bowling pins.) The number representing the total number of cattle then contains 206,545 digits, as was shown in 1880 by A. Amthor. Amthor remarked that a sphere of diameter equal to that of the Milky Way could contain only a small part of this number of animals, even if each were as small as a bacterium.

Most people would consider the problem sufficiently solved for their tastes (or distastes) at this point. But A. H. Bell, a surveyor and civil engineer of Hillsboro, Illinois, succeeded in convincing two of his friends to join him in calculating the first thirty and the last twelve digits of this gargantuan number. But the intermediate 206,503 digits are still unknown. I like to think there is a special place in heaven reserved for people like Mr. Bell and his friends, who work so hard and long (it was a four-year job) for so little tangible reward.

Definition of a Catalyst

A catalyst is familiar to many as a substance which must be present to enable a certain chemical reaction to

take place, but which does not take part in that reaction. It is not always easy to give an example of a catalyst. I have long been fond of the following illustration:

A wealthy Arab died, and his will stipulated that one half of all his horses should go to his oldest son, one third of his horses to his middle son, and one ninth to his youngest son. At the time of his death, the Arab owned seventeen horses, and the sons sorely lamented the thought of carving up some of these fine Arab steeds to comply with their father's will. As they were bewailing their plight, an itinerant Arab with a seedy old nag chanced by, and he offered the following suggestion:

"Look, boys," he said, "let's put my horse in with your father's seventeen. Then we have eighteen. You, son number one, are to get one half of the horses; that is nine. And you, son number two, are to get one third of the eighteen—or six horses. And you, son number three, are to get one ninth, or two. That's a grand total of seventeen, and one is left over. And that's mine. So long, boys!"

The Arab's horse was a catalyst.

Mnemonics for the Value of Pi

The number Pi plays an important role in many branches of mathematics. One of its definitions (and probably the only one familiar to some people) is that it is the ratio of the circumference to the diameter of a circle. The value of π has been calculated to extreme accuracy, and mnemonics are often devised to help remember the digits.

For example, in the sentence, "May I have a large container of coffee?" the number of letters in each word corresponds to the successive integers in the decimal expansion of π.

George Gamow once wrote an article for *Scientific*

*American,** in which he mentioned the value of π, and gave the first five decimals in the expansion as 3.14158. Several months later, a reader wrote to *Scientific American*, chiding Gamow for making a mistake in the fifth decimal place. Gamow answered, apologizing for the error, and explained that it was due to the fact that he was an atrocious speller. It seems that he remembered π from the number of letters in each word of the French poem:

> "*Que j' aime à faire apprendre*
> *Un nombre utile aux sages*
> *Immortel Archimède artiste ingénieur*
> *Qui de ton jugement peut priser la valeur*
> *Pour moi ton problème*
> *A les pareils avantages!*"

He spelled the word "apprendre" with only one "p," and this caused him to recall the value of π as 3.14158. He does not guarantee the accuracy of this poem beyond the first few lines.

A Short History of the Accuracy of Pi

The ratio of the circumference of a circle to its diameter (π) has been known for a long time, and the history of the accuracy to which it is known is fairly interesting.

In the Bible, the value of 3 is used. Archimedes knew that π was less than $3\frac{1}{7}$, but greater than $3\frac{10}{71}$. The value some of us use today, 3.1416, was known at the time of Ptolemy of A.D. 150. F. Vieta, in the sixteenth century, calculated π to ten decimals, and at the end of the sixteenth century, Ludolph van Ceulen calculated 35 decimals. Van Ceulen requested, in his will, that these 35 figures be engraved on his tombstone, and this was done. In Germany the number is still sometimes referred to, in his honor, as

* *Scientific American*, Oct. 1955.

the Ludolphian number. More and more decimals were calculated in the succeeding years, and Shanks, an English mathematician, calculated 707 decimals in 1873. (Poor fellow, he made a mistake in the 528th decimal place, but never found out about it. All succeeding figures are, of course, wrong.)

Shortly after this, Lindemann proved, in 1882, that π was a transcendental number (not the root of a finite algebraic equation with integral coefficients) and therefore could not possibly be a repeating decimal. In spite of this, the Eniac computing machine was used in 1949 to calculate 2,035 decimals of π, and E. T. Bell reports that "recently, some enthusiastic idiots computed π to ten thousand decimals."* The existence of Lindemann's proof of the irrationality of π seems to make the reasons for calculating so many decimals even less attractive than the reason for climbing Everest. But there's no accounting for tastes.

Best of All Possible Mnemonics

One of the branches of physics treats the different probability distributions. One of these distributions is the Poisson distribution, which states that if m happenings occur on the average, then the probability that n will occur is simply $P_m(n) = (m^n e^{-m})/n!$ where e has its usual value of 2.7182 . . ., and $n!$ is n factorial. This expression is fairly important (especially to a graduate student who knows that his mean old professors will expect him to remember it). It is customary, when seemingly arbitrary things have to be learned, to try to devise mnemonics for them. The

* Recently the value of π has been computed to more than 100,000 decimals. (Shanks, David, and Wrench, John W., Jr., "Calculation of Pi to 100,000 Decimals," *Mathematics of Computation*, Jan. 1962, Vol. 16, No. 77, pp. 76–99.) When, oh when, will this madness end?

Poisson distribution has the best of all possible mnemonics*—the first 6 letters of the word *mnemon*ic itself:

(*m* to the *n*, *e* to the minus *m*, *o*ver *n* factorial)

Experimental Determination of Pi

A beautiful theorem, discovered by Euler, relates the sum of the reciprocal powers of the integers to an unusual product involving all prime numbers. The theorem states that:

$$\prod_{\text{all primes}} \frac{1}{1 - p^{-s}} = \sum_{1}^{\infty} \frac{1}{n^s}$$

where s is an arbitrary integer. $\sum (1/n^s)$ is known as $\zeta(s)$. In particular, $\zeta(2)$, that is, the sum of $1/n^2 = \pi^2/6$. It is shown at the end of this chapter that $1/\zeta(2)$ is equal to the probability that two numbers chosen at random are relatively prime. (This is only one of the vast number of unusual and unpredictable relations existing in mathematics.)

R. Chartres did an experiment in 1904 in which he asked 50 of his students to write down five pairs of numbers chosen at random. He found that 154 of the 250 pairs of numbers had no common factor. From this, he could deduce that $1/\zeta(2) = 6/\pi^2 = \frac{154}{250}$. This gives a value of π equal to 3.12. (It is quite apparent that this method of calculation does not have high accuracy as one of its attractions.)

At the Chicago World's Fair in 1893, there was a mechanical implementation of Count Buffon's Needle Problem. In this problem, a needle of length l is repeatedly dropped onto a table on which are ruled parallel lines.

* This mnemonic was devised shortly after I had read, and was impressed by, Voltaire's *Candide*.

These lines are separated by a distance L. The probability that the needle, dropped at random, will intersect one of the parallel lines is equal to:

$$\frac{\text{twice the length of the needle}}{\pi \text{ times the distance between the lines}} = \frac{2l}{\pi L}$$

There were two counters set up in Chicago, one to count the total number of tosses of the needle, the other to count the number of times a line was crossed by the needle. It is said that, after a year of operation, the value of π was approximately 3.16 . . . . This method is also not recommended for its high accuracy. Much better results were reported by Lazzerini. He tossed the needle 3,408 times, and got a value of π with an accuracy of 3×10^{-7} (3.1415929). This accuracy appears to be unreasonably high, since, if there was a normal distribution, one would expect the variation from the true value, divided by the value itself, to be inversely proportional to the square root of the number of trials. Since N, in this case, was 3,408, one would expect the variation in π divided by π to be about $\frac{1}{\sqrt{3408}}$ (between 1 and 2 percent).

There seems to be no doubt that Lazzerini was extremely fortunate in getting such high precision from so few trials.

A Letter in the *Physical Review*

Some people may object that this book, which is not written by a mathematician, pokes fun at mathematics. Why doesn't the author poke fun at his own subject, physics? To show that I am not maliciously biased, I can tell a story about Dr. G. Gamow. He and Dr. R. A. Alpher wrote an article for the *Physical Review*. Before submitting the article, they persuaded Dr. Hans Bethe (pronounced

Bay-tuh) to join them as an author. The article, accordingly, appeared in the *Physical Review* letters,* entitled "On the Abundance of the Elements" by Alpher, Bethe, and Gamow. It seems peculiarly fitting that the origin of the elements should be described by three men whose names are the same as the beginning of the Greek alphabet.

Phi: The "Golden Ratio"

Pi crops up in many places in mathematics when we least expect it. Another constant which has the same pleasant property is known as Phi, or the Golden Ratio. It can be defined in a great many ways, and perhaps the easiest to remember is that it is the only number whose reciprocal is exactly equal to one less than the number.

Thus: $$1/\phi = \phi - 1$$

This leads to: $$\phi^2 - \phi - 1 = 0$$

$$\phi = 1.61803398\ldots$$

To illustrate some of the properties of "ubiquitousness," ϕ is the ratio between any line segment in a five pointed star, and the next smaller segment in that figure. Picture frames, playing cards, and other rectangles we encounter seem to have a propensity for having their sides in the ratio ϕ (*i.e.*, they are Golden Rectangles).

One of the wildest attempts to show the ever-present nature of ϕ was performed by F. A. Lonc, of New York. He measured the heights of sixty-five women, and the heights of their navels. (Why did he choose only women?) He reported that the average of a woman's height divided by the height of her navel was equal to 1.618. It is said that those women whose measurements were not close to this ratio testified to hip injuries or other deformities.

* *Physical Review,* April 1948.

Martin Gardner reports that the same measurements were performed by a group of men (each of whom measured only his own wife) and got a value for this ratio about three percent higher than ϕ. These men can justly claim that they have "high-phi" wives!

Probability That Two Random Numbers Are Relatively Prime, and Euler's Theorem

The fundamental theorem of arithmetic states that each number has a unique decomposition into prime numbers. If two numbers have no common factors, they are said to be relatively prime. Thus, 8 and 11 are relatively prime, even though 8 is not prime itself. The probability that a number is divisible by p is $1/p$, because one out of every p consecutive numbers is divisible by p: *i.e.*, only one in a string of five consecutive numbers is divisible by 5, and only one in a string of seven is divisible by 7. Since the probability that one number, chosen at random, is divisible by p is $1/p$, the probability that two numbers chosen at random have the same common factor p is $1/p^2$, and the probability that they do not have this common factor is $[1 - (1/p^2)]$. The probability that two numbers chosen at random are relatively prime will be the infinite product of terms of the form $[1 - (1/p^2)]$ where the p's range over all the prime numbers. Thus, the probability that two random numbers are relatively prime equals

$$\prod_{\text{all primes}} \left(1 - \frac{1}{p^2}\right)$$

We will now show how to evaluate this product or, more accurately, its reciprocal. We note first that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots x^n + \ldots$$

where the absolute value of x is less than one.

We will take x to be equal to the $1/p^2$ above. Then the reciprocal of the probability is an infinite product of terms, each of which is very similar to the expansion for $1/(1 - x)$.

$$\Pi \frac{1}{1-(1/p^2)} = (1 + p_1^{-2} + p_1^{-4} + p_1^{-6} + \ldots)(1 + p_2^{-2} + p_2^{-4} + p_2^{-6} \ldots) \ldots (1 + p_n^{-2} + p_n^{-4} + p_n^{-6} + \ldots)$$

In this product, every product of primes occurs in unique fashion once and only once, and to the -2 power. It therefore represents the sum of all integers (from 1 to infinity) taken to the -2 power. It can be written as:

$$\Pi \frac{1}{1 - p^{-2}} = 1 + 2^{-2} + 3^{-2} + 4^{-2} + \ldots = \sum_1^\infty \frac{1}{n^2}$$

This last sum is the Zeta (ζ) function, and $\zeta(2)$ is known to be $\pi^2/6$. The probability that two random numbers are relatively prime is the reciprocal of this, namely $6/\pi^2$. If, instead of expanding the infinite product $\Pi[1/(1 - p^{-2})]$ we expand the product $\Pi[1/(1 - p^{-s})]$ then all our arguments could be modified slightly, and we would come out with the result that:

$$\prod_{\text{all primes}} \frac{1}{1 - p^{-s}} = \zeta(s) = \sum_1^\infty \frac{1}{n^s}$$

This surprising result was originally discovered by Euler. (That fellow sure got around, didn't he?)

15: A Slight Electrical Interlude and Some Undigested Thoughts

There are a few problems whose solutions are consistent with the aim of the rest of the book and which are based on the very first principles of electric circuit theory. I have been advised by some people that such problems do not belong in this book. But an equally large and powerful group feels that these problems *must* be included. To satisfy both groups this chapter is included, but labeled so that "electrophobes" can avoid it. For background, one need know that a voltage E (volts) applied across a circuit of resistance R (ohms) will cause a current I (amperes) to flow, and that these quantities are related by Ohm's Law: $E = IR$ or volts = amperes $\times$ ohms.

Two-Dimensional Array of Resistors

Consider an infinite two-dimensional arrangement of one-ohm resistors divided into small squares, as in figure 27. What is the resistance between points A and B?

It is helpful to consider two simpler problems which can be superimposed to give the desired solution. Consider first, that one ampere of current flows in at point A, and

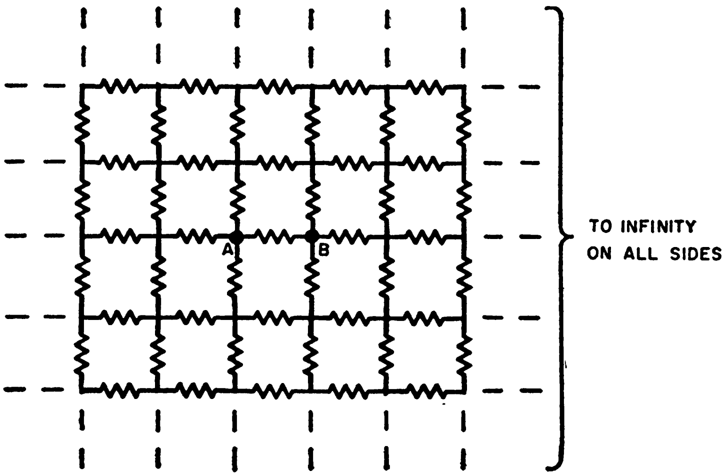


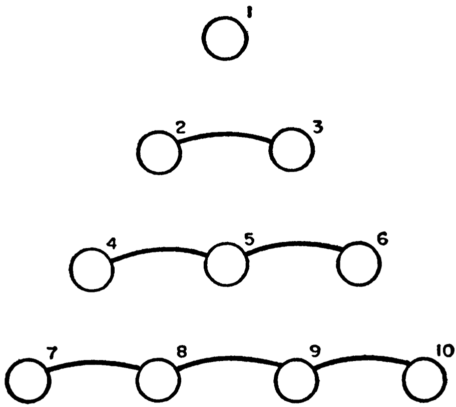
Figure 27

that this current is taken out at infinity (or at a very large circular ring, if you prefer). The current will surely divide into four equal parts, one in each of the four equal resistors leading from A. Now consider that the polarity is reversed, and that the current enters at infinity and is extracted at B. Again, the current must come in equal amounts through the four resistors terminating at B. We can superimpose these two solutions, applying a positive voltage E at A, and a negative E at B. (The circle at infinity will correspond to zero voltage, and the currents flowing in and out will cancel.) Of the current I which enters at A, $I/4$ will flow in the resistor between A and B. And of the current I which is extracted at B, $I/4$ again flows in that resistor. A grand total of $I/2$ flows in that resistance R when these are superimposed, and causes a voltage $E = IR/2$. The effective resistance is thus: $E/I = R/2$.

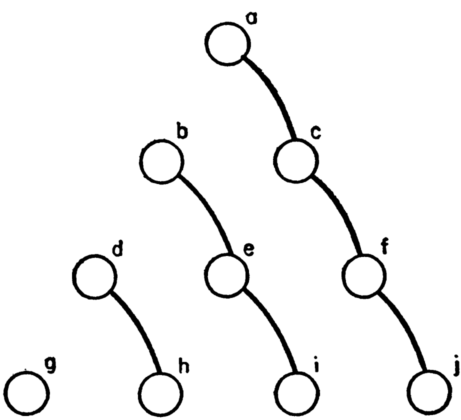
Q.E.D.

Ten Wires Under the River

Consider a ten-wire cable under a river. The wires are not color-coded or marked in any way, and are available for inspection only on either side of the river. It is desired



(a)



(b)

Figure 28

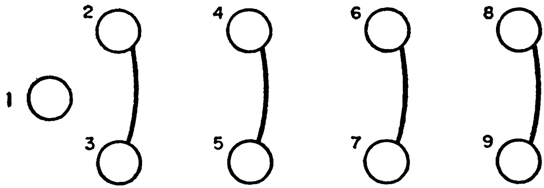
to tag and identify each of the wires. The only implements allowed are a batch of clipleads (with which any wire terminal may be connected to any other terminal, or group of terminals) and an "on-off" continuity meter (with which the presence or absence of a continuous circuit may be determined.) How can all the wires be identified by one man with only one round trip across the river?

There are at least two solutions to this problem. The first one is valid only when the number of wires in the cable is a triangular number (the sum of successive integers starting with one). To solve the problem according to this first method, we consider the wires to be labeled and connected as is shown in figure 28(a).

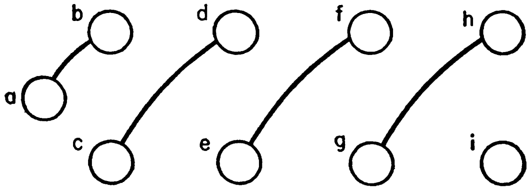
We connect wires in groups of 1, 2, 3, etc. On the other side of the river, we can discover which wires are in groups of 1, 2, 3, etc., but so far only wire 1 is positively identified as being wire *a*. (We know, for example, that the group 4, 5, 6 is the group *d*, *e*, *f*, but we can't separate it any more than that.) Now we connect the wires, as is shown in figure 28(b), and come back to the first side of the river. Then we can again discover that wires are connected in groups of one, two, three, etc., and the fact that each wire is uniquely determined as a member of a group of *m* wires on one side, and of a group of *n* wires on the other side, serves to identify each wire.

If the number of wires is not a triangular number, but is an odd number, a different technique must be used. If there were, for example, nine wires in the cable, we could connect all the wires, save one, in pairs as is shown in figure 29(a).

On the other side, all the wires save one are discovered to be grouped in pairs. Then, since wire 1 is not connected to anything else, it can be identified as *a*. The remaining wires are connected in pairs as is shown in (b). Back on the first side of the river, wire 1 (or *a*) is known to be connected to *b*, and finding which wire is now shorted to number 1 identifies the wire which goes to *b*. The other wire of that



(a)



(b)

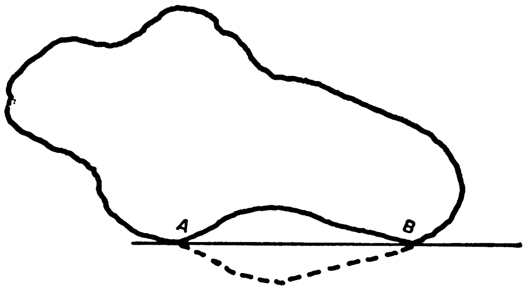
Figure 29

pair is known to be *c*, and the identification procedure continues in this way until all wires are known. If the number of wires is neither odd nor triangular, we can use the odd-wire technique and never connect the extra even wire on either side of the river. This extra wire which was never connected to anything is thus identified.

Perimeter to Enclose Maximum Area

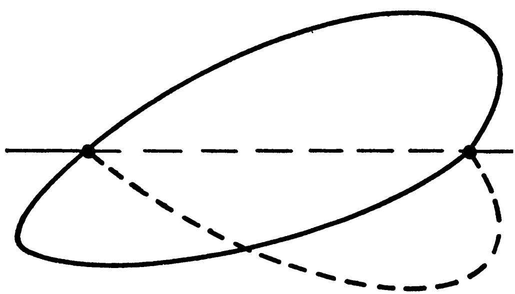
The geometrical figure which encloses the maximum area for a given perimeter is well known to be the circle. The following demonstration provides a simple, but intuitively satisfying, beginning of a proof. Consider a smooth curve of arbitrary shape as is shown by the solid curve in figure 30:

Figure 30



At any indentation, such as the one between points A and B, we can draw a straight line tangent to the curve at those points, and reflect the curve in that line, getting the dotted curve shown. The perimeter of the changed curve is precisely the same as that of the original curve, but the enclosed area has been increased. This reflection procedure can be carried out so long as there is an indentation in the curve, and eventually a smooth curve which has no indentations must result. This curve might be as shown in figure 31:

Figure 31



Two points exactly half the perimeter apart are marked. The line joining them can be called the pseudo-diameter. In general, one of the two areas on opposite sides of this line will be larger. We can replace the smaller area with a mirror image of the larger area, and thus arrive at an area larger than the original, but still of the same

perimeter. If the smooth curve is not perpendicular to the pseudo-diameter, then there will be a dimple, or indentation, in the new curve. And then the process of increasing the area, but not the perimeter, can be continued as before. Only in the case where the curve is everywhere perpendicular to the pseudo-diameter can we no longer increase the area without increasing the perimeter. But this latter case defines a circle. So the circle must be the curve we sought.

Major shortcomings of this demonstration are that it is not constructive, and it does not adequately deal with angular figures, which may arise from reflection in the pseudo-diameter. The beginning of the proof is presented, however, in the hope that some reader may be able to continue along these lines, and arrive at a proof of this chestnut in rigorous fashion.

16: Left-Overs—from Greeks to Computers

A Computer Correlation

The first digital computer was unveiled right after World War II. In the succeeding years, there was a period when new computers were appearing on the market faster than they were being used to solve problems. During that blissful era, a computer could be found for almost any interesting problem. In particular, one fellow fed vast amounts of astronomical and geophysical data into a machine, and asked the machine to correlate this data.

The machine did its bidding, and came up with many interesting correlations. One of them was that every time the fifth moon of Jupiter was in opposition to our moon, it rained in Montreal, Canada. The fellow who was running this program realized how very foolish it was to think that the position of Jupiter's moon relative to our moon could affect the weather in Montreal. It was more likely, he thought, that the rain in Montreal, Canada, *caused* the fifth moon of Jupiter and our moon to go into opposition.

Translation of Languages by Computer

Computers are very much in the news today. Many self-appointed prophets would have us believe that the world will shortly become unrecognizable by our present standards. There is little doubt that many new uses for computers will be discovered, and that life will be changed.

But the indiscriminate application of computers might have drastic consequences—such as is indicated by the following story:

An engineer had devised a program for a computer to translate anything between any two languages on earth. He reported this development at a technical meeting, and to demonstrate the power of his program, asked the audience to suggest a phrase for the machine to translate. Someone suggested "Out of sight, out of mind." He entered this into the computer, and then asked the audience for a language into which he could translate the phrase. Another person suggested Russian, and the computer was instructed to translate "Out of sight, out of mind" into Russian. The machine whirred for a few moments and announced that the required translation was

СЛЕПОЙ ИДИОТ.

Unfortunately, no one in the room understood Russian. So no one knew whether it was a good translation or not. Finally, someone had the brilliant idea that this Russian could be fed into the machine with instructions to translate it back to English. This was done, and the machine printed that "Out of sight, out of mind," translated into Russian and back into English, resulted in the phrase "Blind idiot!"

This story can also be told where the sentence suggested is "The spirit is willing, but the flesh is weak." This, translated into Russian, and *that* Russian translated back into English, becomes "The whiskey's O.K., but the meat is lousy."

The difficulties that computers encounter in translating languages are apparent.

Fair Division of Cake Among *N* People

If there is a piece of cake to be shared by two people, it is surely fair to let either person cut the cake and allow the other person to choose his piece first. In this way,

neither person has any right to complain. Generalize this fair procedure for N people, so that no one can complain, regardless of collusion among any of the N people.

Any one of the N people is asked to cut for himself a piece of cake with which he would be satisfied. The remaining people are then asked if any of them think the first man has too large a piece. If there are any who do, one of them is asked to cut from this piece of cake a slice such that he (the person cutting) would be satisfied with the piece left. If no one feels that this piece of cake is too large, then this second man may have that piece of cake. And he cannot justly complain. If anyone, or any group, does object, the procedure can be carried on until no one does. The entire procedure is then repeated with $(N - 1)$ people, $(N - 2)$ people, etc., until everyone has his cake. The advantage of doing this problem with orange juice instead of cake should be considered.

The Trolley Cars: An Apparent Violation of Probability

A young man has two girl friends who live in diametrically opposite directions from his house, and whom he normally sees about an equal number of times each month. The trolley line which goes to both of their houses passes in front of his house. Since he has an equal regard for both of them, he decides to leave his house at completely random times, and to take the first trolley that comes along. In this way, he thinks he will continue to see each of them an approximately equal number of times, since the trolleys go in each direction with the same regularity. He follows this regime for a few months and, much to his consternation, he finds that the laws of probability are apparently being violated. He has gone to see one girl friend five times as often as he has gone to see the other! How can this be?

In order to see how the laws of probability can apparently be violated, it is convenient to picture the track along which the trolleys run, as is shown in figure 32a:

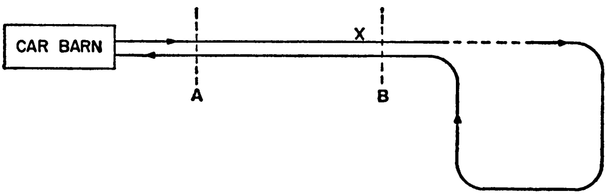


Figure 32a

The trolleys run with perfect regularity in both directions, and because of this, there will be certain places where two cars going in opposite directions *always* pass each other. These are at points marked A and B. Assume further, that the young man lives at the point marked x, which is 1 minute from B, and 11 minutes from A. We start our observation at the moment when the cars pass each other, and if the man appears at x within the next minute, he will meet a car going from B to A. If he appears during the next 10 minutes after that, he will have missed the car going from B to A, but will be in time for the car going from A to B. One minute after the car from A passes x, it will have reached B; the cycle will start over again. If the man appears during this last minute, he will have missed the car from A to B, but can wait for the next car from B to A. Table 14 shows the situation during the entire cycle:

Time	*Situation*
0–1 min.	next car will be from B
1–11 min.	next car will be from A
11–12 min.	next car will be from B

Table 14

Thus, the next car will come from B only during two minutes of each cycle. But during ten minutes of the cycle, the next car will be from A. If the man comes out at random times, he is five times more likely to go toward B than he is to go toward A.

Napoleon's Problem

A curiosity which demands inclusion in a book such as this is sometimes known as Napoleon's Problem (I've never found out why). This problem asks for the construction *with compass alone* of two points which, with a given pair of points, would be at the four corners of a square. This problem is ridiculously easy with ruler *and* compass, but can be tantalizingly difficult with compass alone.

Figure 32b shows the two given points labeled A and B. With radius AB, a circle is drawn about B, and an arc around A. Point C, which is diametrically opposite A, is found by the usual method of dividing a circle into six

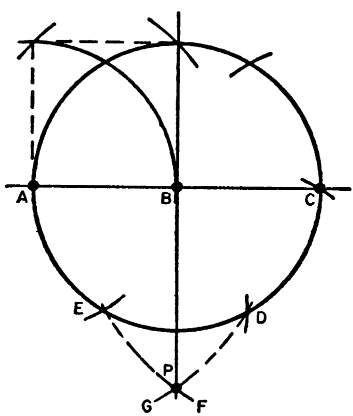


Figure 32b

parts.* Then, with radius CE, the arc EF is described about C, and likewise, with radius AD (the same as CE), the arc DG is described. These two arcs intersect at point P and the line BP is perpendicular to AC. Triangle AEC is a right triangle because it is inscribed in a semicircle, and the right angle is angle AEC. Thus, we have the equation $(AE)^2 + (EC)^2 = (AC)^2$, or in terms of r we can write:

$$r^2 + EC^2 = (2r)^2 = 4r^2$$

$$EC = AB\sqrt{3} = r\sqrt{3}$$

Two additional arcs as in figure 32b can then be drawn because the length BP is the diagonal of the desired square.

The Mohr-Mascheroni Construction

In 1797 Mascheroni showed that all constructions possible with ruler and compass were possible with compass alone. A Danish mathematician named Mohr had done approximately the same thing in 1672. These constructions are now known as the Mohr-Mascheroni constructions. It is understood that straight-line figures cannot be drawn, but the points defining these figures can be constructed.

To prove that all constructions possible with ruler and compass are possible with compass *alone*, it is sufficient to show that a compass alone can be used to determine any point which could have been determined with ruler and compass. With both instruments we can determine the intersections between:

* On a circle of radius r, exactly six consecutive and contiguous chords of length r can be drawn. Each of these chords with two of the radii of the circle will form an equilateral triangle. Each angle of the equilateral triangle is known to be 60°, and six 60° angles are equal to 360°, or a full circle. A circle can thus be easily divided into six sections with a compass, by stepping off the radius six times around the circumference.

(*a*) two circles

(*b*) a circle and a straight line which does not pass through the center of the circle

(*c*) a circle and a straight line which does pass through the center of the circle

(*d*) two straight lines

In all of these cases, the straight line is defined by two points on it.

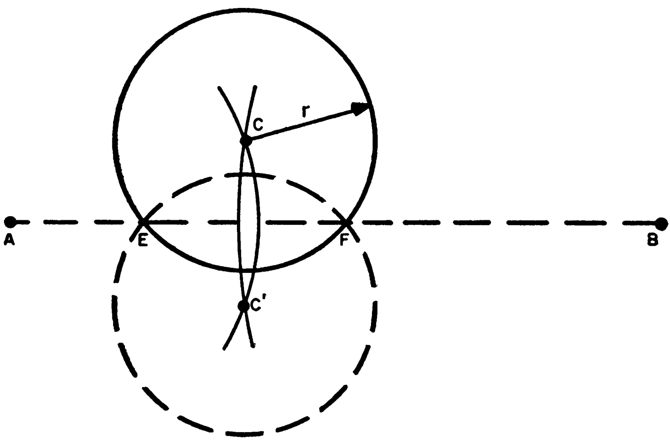


Figure 33

Construction (*a*) above is trivial. The circles are drawn and the points of intersection are evident. Construction (*b*) is shown in figure 33. The straight line is defined by points A and B. The circle is centered at C and is of radius r. The point C', which is the image of point C in the line AB, is found by drawing arc CC' centered at A and the other arc CC' centered at B. A circle of radius r centered at C' will intersect C in the two required points, E and F.

Construction (*c*) is more complex; it is outlined in figure 34:

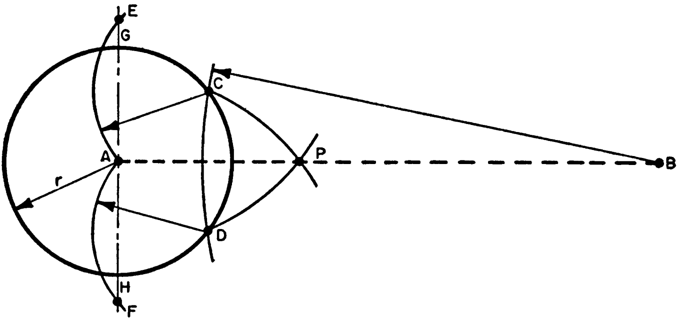


Figure 34

The line AB is indicated by the dotted line in figure 34, and the circle of radius r is centered at A. With arbitrary radius BC, arc CD is drawn centered at B. The problem is to bisect arc CD. Arc AE is drawn centered at C with radius r, and arc AF is centered at D with the same radius. Length CD is measured on arcs AE and AF, defining points G and H. Line AG is parallel to CD as is AH, because both $AGCD$ and $AHDC$ are parallelograms. Arcs DP, centered at G, and CP, centered at H, are drawn, intersecting at P. Length GD is the hypotenuse of a right triangle whose base is $\frac{3}{2}\,CD$ and whose altitude is

$$\sqrt{r^2 - (CD/2)^2}$$

The length GD is thus:

$$\sqrt{r^2 - \tfrac{1}{4}\,(CD)^2 + \tfrac{9}{4}\,(CD)^2} = \sqrt{r^2 + 2(CD)^2}$$

The length AP is then:

$$\begin{aligned}\sqrt{(GD)^2 - (AG)^2} &= \sqrt{r^2 + 2(CD)^2 - (CD)^2} \\ &= \sqrt{r^2 + (CD)^2}\end{aligned}$$

The point x which is the midpoint of arc CD is a distance

$\sqrt{r^2 + (CD)^2} = AP$ from point G and thus can be constructed.

Construction (*d*) is by far the most difficult of the four. Figure 35 shows the four points A, B, C, and D, and the lines connecting them are shown dotted:

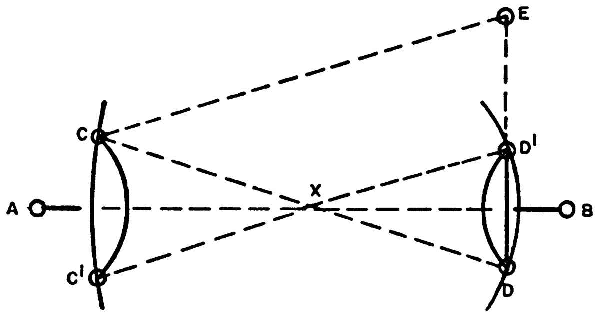


Figure 35

Points C' and D' are the images of C and D in line AB. Point E is now constructed to be the fourth point in parallelogram $CC'D'E$. Since CC' is perpendicular to AB, $D'E$ is also. And since DD' is also perpendicular to AB, $ED'D$ must be a straight line. In triangle DCE, $DX/DC = DD'/DE$. The length DX is the only unknown in this equation. It is called the fourth proportional. It can be constructed as shown in figure 36.

The circles of radius DD' and DE (both known) are drawn with the same center. At an arbitrary point P on the smaller circle, the arc RS is drawn with radius DC. The arc TU is drawn about S with the same radius DC. The length RT is the fourth proportional, *i.e.*, is the length DX which we sought to construct. This is so because $\angle ROT = \angle POS$ ($\angle POR = \angle SOT$ and $\angle ROS$ is added to both of them). Chords PS and RT subtend the same size angle at the center and they are, therefore, proportional to the radii of their circles. Having length DX,

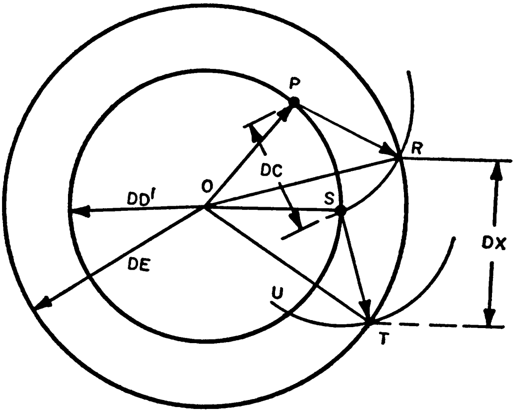


Figure 36

we go back to figure 35 and describe arcs DX and $D'X$, thus defining point x. The construction is now complete.

And since the ingredients of a construction with ruler and compass have now been done with compass alone, the proof of the Mascheroni Construction is complete.

"Mohr" of these Mascheroni constructions are described in a booklet by A. N. Kostovskii.*

Random Chord of a Circle

What is the probability that a random chord of a circle is longer than one side of the inscribed equilateral triangle?

A random chord will be determined by two random points on the circumference of the circle. An inscribed equilateral triangle can be rotated inside the circle so that one of its apices coincides with one of the points. Then, if the second point falls between the other two apices, the random chord will be longer than a side of the triangle.

* Kostovskii, A. N., *Geometrical Construction Using Compass Only*. New York: Blaisdell Scientific Publications, 1961, vol. 4 of Popular Lectures in Mathematics Series.

The available circumference to get the longer chord is $\frac{1}{3}$ of the full circle, so the probability must be $\frac{1}{3}$.

But there is another way to solve this problem. We can say that a random chord is determined by a point on a randomly oriented radius vector of the circle. Wherever this point falls, we draw a line perpendicular to the radius vector, and all possible chords can be represented this way. Only if the randomly chosen point is within $\frac{1}{2}$ of the radius of the circle will the random chord be longer than the side of the triangle. The random point can fall anywhere along the radius, and since only the centermost half yields a favorable result, the probability that the chord is longer must be $\frac{1}{2}$.

But hold! What have we here? (Enter guards with clanking armor to remove author.) Two different answers for the same mathematical problem? Let's try a third way. And if the third way comes out the same as either of the other two, then we will at least have the weight of probability on our side. A random point in the interior of the circle will determine a random chord because we can draw a line from the point to the center, and then a perpendicular to that line determines the chord. But the point must be within a circle of radius $r/2$ in order that the chord be longer than the side of the triangle. The probability of getting a randomly chosen point within the center circle of $\frac{1}{2}$ the full circle radius is obviously proportional to the area, and hence the required probability is $\frac{1}{4}$.

This demonstration shows rather clearly that words like "a randomly chosen chord" must be carefully considered and defined.

Description of Greek Names for Numbers

The Greeks had many names for numbers. Numbers were not simply odd or even, or large or small, but they were deficient, perfect, sociable, friendly, abundant, square, etc.

Perfect numbers are those for which the sum of all the divisors of the number (including one, but excluding the number itself) is equal to that number. Thus, 6 is perfect because its factors, 1, 2, and 3, have the sum 6. The factors of 28, 1, 2, 4, 7, and 14, have the sum 28, and so 28 is also perfect. Perfect numbers are very well understood, and there is a formula $2^{n-1}(2^n - 1)$ where $(2^n - 1)$ is prime for all the perfect numbers known so far. (A tantalizing question is, "Are there any *odd* perfect numbers?")

Deficient numbers are those in which the sum of the divisors is *less* than the number itself, and abundant numbers are those in which the sum is greater. Examples of deficient and abundant numbers are not difficult to find.

Friendly (or amicable) pairs of numbers are those in which the sum of the divisors of each number is equal to the other number, as for example, 284 and 220. The divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110—which total 284; the factors of 284 add up to 220.

In sociable chains the sum of the divisors of each number is equal to the next number, and the sum of the divisors of the last number is equal to the first number. (It requires what the Germans call "sitzfleisch" to discover such chains, but number theorists are apparently willing to devote time to such problems.) An example of a sociable chain is 14,288, 15,472, 14,536, 14,264, 12,496. (I can't bring myself to verify this, or to ask any of my friends to do so.)

Some Geometricians Are Masochists

The usual picture of a mathematician is that of a shy individual, often with glasses and a receding forehead (or a large scalp.) Geometricians, in particular, have peculiar ways of torturing themselves. It is known, for example, that Euclid made many geometrical constructions using only a straightedge and a compass. But some geometricians go still further—they try to use less than the standard straightedge and compass.

For example, Mohr and Mascheroni showed how many constructions could be accomplished with a compass alone. Steiner* showed that Euclid's constructions could be accomplished with a single circle and a straightedge. Desargues' Theorem and the Wilkes-Gordon Constructions show how the straightedge alone can be used to accomplish certain difficult constructions.

A problem was recently reported in which one is required to find the perpendicular bisector of the line joining 2 points, using only a tri-square. A tri-square is an implement which can draw straight lines and *erect* a perpendicular at any given point on a line, but it *cannot* be used to *drop* a perpendicular to a line.

There have been so many examples in which seemingly idle amusements turned out subsequently to have great and useful applications, that it would be foolhardy to criticize geometricians for limiting the tools available to them. (But only some mathematicians consider practicality to be a desirable feature. For example, it is said that Kummer, the mathematician, was once asked which of all his discoveries he appreciated most. He replied that he appreciated ideal numbers most, because they had not yet been soiled with practicality.)

Ancient, Admirable Calculations

Figure 37 shows a pair of intersecting cylinders, both of the same radius r. What is the volume common to both cylinders?

This problem can easily be solved by integral calculus, but the answer was known to Archimedes, who antedated the discovery of calculus by many years. What methods did he use?

We don't know the method used by Archimedes, but it may have been similar to the following:

* Courant and Robbins (*loc. cit.*), p. 197.

Consider a sphere of radius r to be at the center of the volume common to both cylinders. If we make a series of slices parallel to the plane of both cylinders, we will find that each thin section of the volume common to both cylinders is a square; and that a portion of the sphere is present in each one of these slices as a circle inscribed in that square. Each slice, then, consists of a square from the

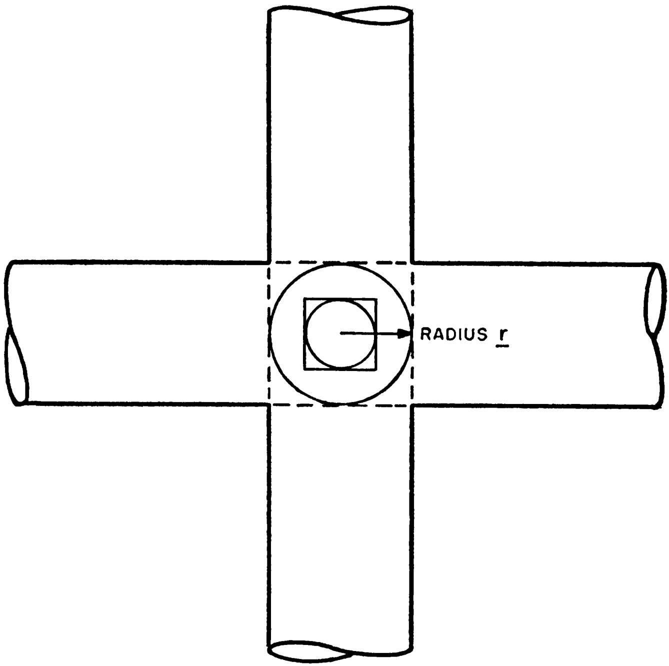


Figure 37

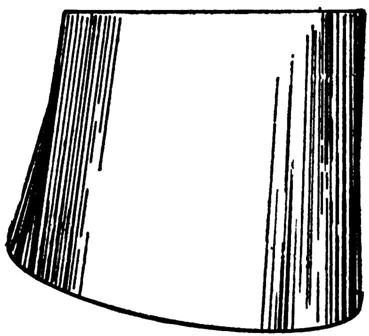
common volume and an inscribed circle from the sphere. The area associated with the sphere in each slice in πa^2, and the area associated with the common volume is $(2a)^2$, where a represents the radius of the inscribed circle. The ratio of area in each slice belonging to the common volume

to that belonging to the sphere is $4a^2/\pi a^2$. The total volume, which is the sum of all these slices, will be in the same ratio of $4/\pi$. So the common volume must be

$$\frac{4}{\pi}\left(\frac{4}{3}\,\pi r^3\right) = \frac{16}{3}\,r^3.$$

A similar problem asks for the volume of the solid shown in figure 38. The solid has a circular base, and a "straight-line top." The sides can then be considered to be straws starting at the top line and going to the circumference of the circle at the bottom but always perpendicular to the line at the top. What is the volume of this figure?

Figure 38



Again, calculus can be used to get the answer, but we ask here for the answer without the aid of calculus. Figure 39 shows how to calculate this volume. Again, we slice the figure in a plane perpendicular to the line at the top. Each thin slice will have the shape of a triangle. If we compare the given figure with a right circular cylinder of the same base and same height, each slice of figure 39 is exactly one-half of the comparable slice of the right circular cylinder. The entire volume of figure 38 is exactly half of the volume of the comparable right circular cylinder. The volume is thus equal to $\frac{1}{2}\pi r^2 L$.

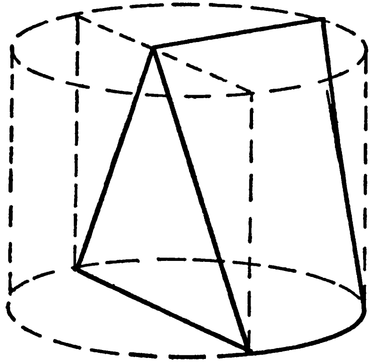


Figure 39

Fermat's Challenge

The mathematicians in a bygone era had one very interesting characteristic which seems to have disappeared. They used to challenge each other or all mathematicians of a particular nationality, to do a particular problem. For example, in 1657 Fermat posed the problem of finding a cube which, when increased by each of its divisors, is equal to a square. One answer is $7^3 + 1 + 7 + 7^2 = 400 = 20^2$.

On another occasion, he challenged the English mathematicians to find all integral solutions of the equation: $x^2 - ay^2 = 1$.* This challenge, incidentally, was correctly answered by Lagrange in 1767. It seems a pity that this method wasn't developed to provide a means for settling problems between nations.

Dance of the Bugs

We have calculated the probability that two numbers chosen at random are relatively prime. Nowhere in this proof was it necessary for us to define exactly what we

* Beiler, Albert H., *Recreations in the Theory of Numbers.* New York: Dover Publications, pp. 248 ff.

mean by a random number. But in spite of this, the calculation of the probability was performed. (I even think it would satisfy the mathematical purist.) The present problem asks for a determination of the length of a very peculiar curve. We can solve this problem without determining the shape of the curve.

Consider a square 10 inches on a side, as is shown in figure 40. A bug waits at each corner of the square and at a given signal (these are trained bugs) each one begins to walk toward the one in front of it, as is shown by the arrows. As each bug moves, the direction of the one behind him changes, because the bugs always move toward the one in front. They move at a velocity of one inch per minute. How long does it take before the bugs collide with each other in the center of the square, and how far has each one walked?

In this problem also, we have an apparent dearth of information. To solve it, we need recognize only that the travels of bug B are *always* at right angles to the path from bug A to bug B, and so cannot affect the distance from A to B. Then it is only A's walking that has any effect on the distance from A to B, and only B's travels affect the distance from B to C. Since the bugs start 10 inches apart, they can travel 10 inches before a central collision. Each bug must travel a distance of 10 inches,

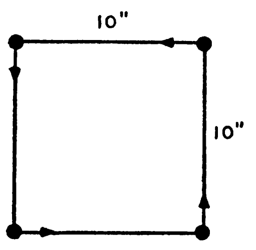


Figure 40

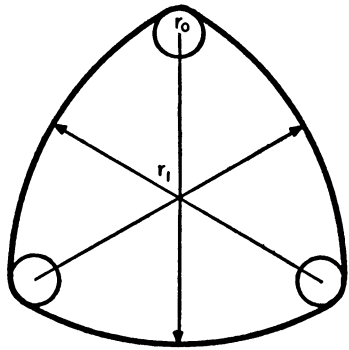
and, at a velocity of one inch per minute, this will require 10 minutes. But the shape of the curve that the bugs travel is still not determined.

The Swiss Mountain Moving Company, Ltd.

(I leave it to the reader to decide whether the third word in the title is used in an adverbial or nominal sense.)

A group of movers were accustomed to using rollers on which to transport heavy objects, but these rollers kept rolling downhill. The men tried using square rollers, but

Figure 41



found the up-and-down motion of the object being moved extremely objectionable. They consulted an oracle to find out if it was possible to design a roller which would not roll downhill of its own accord, but on which a heavy object could be moved smoothly. The oracle (who was a toothless old hag, as are all respectable oracles) suggested that there were many curves which had a "constant diameter" (the diameter of any closed curve is defined as the longest straight line that can be drawn inside the figure) but which were not circular. The fact that they weren't circular would prevent them from rolling downhill. And the fact that they

were of constant diameter would allow objects to roll smoothly.

What could have been the cross-sectional shape of these rollers?* The drawing in figure 41 shows one such curve. At each of the corners of an equilateral triangle, a small circle of radius r_0 is drawn. And from each of these points, an arc of radius r_1 is drawn tangent to the other two circles. A roller with this cross section will not roll down a gentle hill. But an object placed on top of such a roller will remain at a constant distance above the ground.

* A West Coast electronics firm uses such rollers as an advertising gimmick.

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