Chapter 1

Basic Mathematical Objects

1.1. (a) $\{(-1)^n n \mid n \in \mathcal{N}\}$

(b) The denominator of an element can be any positive power of 2, and the numerator can be any positive integer less than the denominator. One formula is therefore

$$\{\frac{m}{2^j} \mid j > 0 \text{ and } 0 < m < 2^j\}$$

(c) $\{1^n 0^n \mid n > 0\}$
(d) $\{\{n\} \mid n \in \mathcal{N}\}$
(e) $\{\{j \in \mathcal{N} \mid 0 \le j \le n\} \mid n \in \mathcal{N}\}$
(f) $\{\{j \in \mathcal{N} \mid 0 \le j < 2^n\} \mid n \in \mathcal{N}\}$

1.4. (a)

$$\begin{aligned} A - (A - B) &= A \cap (A \cap B')' \\ &= A \cap (A' \cup B) \\ &= (A \cap A') \cup (A \cap B) \\ &= \emptyset \cup (A \cap B) \\ &= A \cap B \end{aligned}$$

(b)

$$\begin{aligned} A - (A \cap B) &= A \cap (A \cap B)' \\ &= A \cap (A' \cup B') \\ &= (A \cap A') \cup (A \cap B') \\ &= \emptyset \cup (A \cap B') \\ &= A \cap B' \\ &= A - B \end{aligned}$$

(c)

$$\begin{aligned} (A \cup B) - A &= (A \cup B) \cap A' \\ &= (A \cap A') \cup (B \cap A') \\ &= \emptyset \cup (B \cap A') \\ &= B \cap A' = B - A \end{aligned}$$

(d)

$$\begin{aligned} (A - B) \cup (B - A) \cup (A \cap B) &= (A - B) \cup (A \cap B) \cup (B - A) \\ &= (A \cap B') \cup (A \cap B) \cup (B - A) \end{aligned}$$

$$
\begin{aligned}
&= (A \cap (B' \cup B)) \cup (B - A) \\
&= (A \cap U) \cup (B \cap A') \\
&= A \cup (B \cap A') \\
&= (A \cup B) \cap (A \cup A') \\
&= (A \cup B) \cap U \\
&= A \cup B
\end{aligned}
$$

(e) $(A' \cap B')' = (A')' \cup (B')' = A \cup B$
(f) $(A' \cup B')' = (A')' \cap (B')' = A \cap B$
(g)

$$
\begin{aligned}
A \cup (B \cap (A - (B - A))) &= A \cup (B \cap (A \cap (B \cap A')')) \\
&= A \cup (B \cap (A \cap (B' \cup A))) \\
&= A \cup (B \cap ((A \cap B') \cup (A \cap A))) \\
&= A \cup (B \cap ((A \cap B') \cup A)) \\
&= A \cup (B \cap (A \cup (A \cap B'))) \\
&= A \cup (B \cap A) \\
&= A \cup (A \cap B) \\
&= A
\end{aligned}
$$

(h)

$$
\begin{aligned}
A' \cup (B - (A \cup (B' - A))) &= A' \cup (B - (A \cup (B' \cap A'))) \\
&= A' \cup (B - ((A \cup B') \cap (A \cup A'))) \\
&= A' \cup (B - (A \cup B')) \\
&= A' \cup (B \cap (A \cup B')') \\
&= A' \cup (B \cap (A' \cap (B')') \\
&= A' \cup (B \cap (A' \cap B)) \\
&= A' \cup (A' \cap B) \\
&= A'
\end{aligned}
$$

1.6. In a 2-set Venn diagram, the four disjoint regions are $A - B$, $A \cap B$, $B - A$, and $A' \cap B'$. Let us denote these R_1, R_2, R_3, and R_4, respectively. Then in each part of the exercise, the given equation can be rewritten so that each side is the union of one or more of the R_i's. One way to simplify the equation is just to observe that because any two of the R_i's are disjoint, any R_i that appears on one side of the equation and not the other must be empty, because any $x \in R_i$ would be an element of one side of the equation but not the other. Furthermore, the statement that certain R_i's are empty is the same as the statement that their union is empty (again using the fact that they are all disjoint). In several of the parts, the statement that the union is empty can be simplified.

(a) This equation says $R_1 \cup R_3 = R_1 \cup R_2$. It is therefore equivalent to $R_2 \cup R_3 = \emptyset$, or $B = \emptyset$.

(b) $R_1 \cup R_3 = R_1$, or $R_3 = \emptyset$. This means $B - A = \emptyset$, which is the same as $B \subseteq A$.

(c) $R_1 \cup R_3 = R_1 \cup R_2 \cup R_3$, or $R_2 = A \cap B = \emptyset$.

(d) $R_1 \cup R_3 = R_2$, or $R_1 \cup R_2 \cup R_3 = \emptyset$, or $A \cup B = \emptyset$, or $A = B = \emptyset$.

(e) $R_1 \cup R_3 = R_3 \cup R_4$, or $R_1 \cup R_4 = \emptyset$, or $B' = \emptyset$, or $B = U$.

1.7. (a) Two correct formulas are

$$(A \cup B) \cap (A \cap B)' \text{ and } (A \cap B') \cup (A' \cap B)$$

(b) Two correct formulas are

$$(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C) \text{ and } (A \cup B \cup C) \cap ((A \cap B) \cup (A \cap C) \cup (B \cap C))'$$

(c) Two correct formulas are

$$(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C) \cup (A' \cap B' \cap C') \text{ and } (A' \cap B') \cup (A' \cap C') \cup (B' \cap C')$$

(d) Two correct formulas are

$$(A \cap B) \cup (A \cap C) \cup (B \cap C) \cap (A \cap B \cap C)' \text{ and } (A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C)$$

(e) $(A \cup B \cup C) \cap (A \cap B \cap C)'$

1.8. (a) C_{10}

(b) D_1 (or C_1)

(c) C_1 (or D_1)

(d) D_{10}

(e) $\mathcal{R}$

(f) D_1 (or C_1)

(g) C_1 (or D_1)

(h) $x < 1/n$ for every $n \geq 1$ if and only if $x \leq 0$. Therefore, the answer is $\{x \mid x \leq 0\}$.

(i) $\mathcal{R}$

(j) $\emptyset$ (because no real number is less than n for every integer n)

1.10. The elements of $2^{2^{\{0,1\}}}$, or $2^{\{\emptyset,\{0\},\{1\},\{0,1\}\}}$, are (one per line)

$\emptyset$,
$\{\emptyset\}$,
$\{\{0\}\}$,
$\{\{1\}\}$,
$\{\{0,1\}\}$,
$\{\emptyset, \{0\}\}$,
$\{\emptyset, \{1\}\}$,
$\{\emptyset, \{0,1\}\}$,
$\{\{0\}, \{1\}\}$,

$\{\{0\},\{0,1\}\}$,
$\{\{1\},\{0,1\}\}$,
$\{\emptyset,\{0\},\{1\}\}$,
$\{\emptyset,\{0\},\{0,1\}\}$,
$\{\emptyset,\{1\},\{0,1\}\}$,
$\{\{0\},\{1\},\{0,1\}\}$,
and $\{\emptyset,\{0\},\{1\},\{0,1\}\}$ (16 elements in all)

1.12. All are true.

1.13. In each case, a simpler statement is given.
(a) $\neg p$
(b) *true* (i.e., the statement is a tautology)
(c) $p \wedge q$
(d) q
(e) q
(f) q

1.14. In the case where p is false and q is true, $(p \wedge (p \rightarrow q))$ is false, so the conditional $r \rightarrow s$ must be true when r is false and s is true. In the case where p is true and q is false, $(p \wedge (p \rightarrow q))$ is false, so the conditional $r \rightarrow s$ must be true when r and s are false.

1.15. (a) $(m_1 < m_2) \vee ((m_1 = m_2) \wedge (d_1 < d_2))$
(b) $(m_1 \leq m_2) \wedge ((m_1 < m_2) \vee (d_1 < d_2))$. The second clause in this statement could also be written $((m_1 = m_2) \rightarrow (d_1 < d_2))$.

1.17. (a) neither
(b) contradiction
(c) neither
(d) tautology
(e) contradiction

1.18. $\forall x(\exists y(\exists t(L(x,y,t))))$ This is the most likely interpretation. However, if we interpret the statement to mean that there is a single time at which everybody loves somebody, the correct statement is $\exists t(\forall x(\exists y(L(x,y,t))))$.

1.19. "You can fool all the people some of the time" might be translated either $\exists t(\forall x(F(x,t)))$ or $\forall x(\exists t(F(x,t)))$, and similarly, there are two nonequivalent translations of "you can fool some of the people all the time." Therefore, there are at least four interpretations of the statement.

1.20. (a) True for $(0,1)$, false for $[0,1]$.
(b) True for any domain that is a subset of $\mathcal{R}$.

(c) False for any domain that is a subset of $\mathcal{R}$.
(d) False for $(0, 1)$, true for $[0, 1]$.

1.21. (a) B has at least n elements.
(b) B has at most n elements.

1.22. (a) f_a is one-to-one. The range of f_a is $\{x \in \mathcal{R}^+ \mid x \geq a\}$.
(b) d is one-to-one and onto.
(c) t is one-to-one. The range of t is $\{n \in \mathcal{N} \mid n \text{ is even}\}$.
(d) g is not one-to-one, but it is onto.
(e) p is one-to-one. The range of p is the set of all numbers x such that for some nonnegative integer n, $2n \leq x < 2n + 1$. In other words, the range is

$$\bigcup_{n=0}^{\infty} [2n, 2n+1) = [0,1) \cup [2,3) \cup [4,5) \cup \dots$$

where $[a, b) = \{x \mid a \leq x < b\}$.
(f) i is not one-to-one. The range of i is $2^{[0,1]}$, the set of subsets of $[0, 1]$.
(g) u is not one-to-one. The range of u is $\{S \subseteq \mathcal{R} \mid [0,1] \subseteq S\}$, the set of supersets of $[0, 1]$.
(h) m is not one-to-one. The range of m is the interval $[0, 2]$.
(i) M is not one-to-one. The range of M is the set $\{x \mid x \geq 2\}$.
(j) s is one-to-one. The range of s is $\{x \mid x \geq 2\}$. (Note that $s(x) = x + 2$ for every x.)

1.23. f is onto and g is one-to-one.

1.24(a) For $y \in B$, if there are several x's in A with $f(x) = y$, the formula $f(g(y)) = y$ will hold as long as $g(y)$ is defined to be any of these x's. So, for example, the formulas $g(7) = 2$, $g(8) = 4$, $g(9) = 3$, and $g(10) = 6$ define a function g that works, but $g(7)$ could also be 18, $g(9)$ could be 12, etc. In general, if f is onto, f will have at least one such "right inverse" g; if f is onto and not one-to-one, there will be more than one.
(b) The given information determines the values f must have on the elements 2, 6, 7, and 18. ($f(2)$ must be 9, since $f(g(9))$ is supposed to be 9; etc.) The values of f at the other three elements of A are arbitrary, and so there are many functions that would work.

1.25. (a) $g \circ d(x) = \lfloor 2x \rfloor$
(b) $t \circ g(x) = 2\lfloor x \rfloor$
(c) $t \circ t(x) = 4x$
(d) $d \circ f_a(x) = 2(x + a)$
(e) $f_a \circ d(x) = 2x + a$
(f) $g \circ f_a(x) = \lfloor x + a \rfloor$
(g) $u \circ i(A) = (A \cap [0, 1]) \cup [0, 1] = [0, 1]$
(h) $i \circ u(A) = (A \cup [0, 1]) \cap [0, 1] = [0, 1]$

1.26. (a) $f^{-1}(x) = x$

(b) Since every y in the codomain is positive, the equation $1/(1+x) = y$ can be rewritten $1+x = 1/y$, and this is the same as $x = 1/y - 1$. Clearly, for each y in the codomain, there is at most one x in the domain for which this equation holds, and thus f is one-to-one. Moreover, any y in the codomain is ≤ 1, so $1/y \geq 1$, so $1/y - 1 \geq 0$; therefore, the solution x is in fact in the domain of f, so that f is onto. The solution x to this equation is $f^{-1}(y)$, so we obtain $f^{-1}(y) = 1/y - 1$.

(c) For any two real numbers a and b, the equations $x + y = a$, $x - y = b$ have the unique solution $x = (a+b)/2$, $y = (a-b)/2$. Therefore, f is one-to-one and onto, and $f^{-1}(a, b) = ((a+b)/2, (a-b)/2)$.

1.28. (a) not reflexive, symmetric, not transitive.

(b) reflexive, not symmetric, transitive.

(c) not reflexive, but both symmetric and transitive.

1.29. In each case, a relation with as few pairs as possible is given.

reflexive	symmetric	transitive	
true	true	true	$\{(1,1),(2,2),(3,3)\}$
true	true	false	$\{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$
true	false	true	$\{(1,1),(2,2),(3,3),(1,2)\}$
true	false	false	$\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$
false	true	true	$\emptyset$
false	true	false	$\{(1,2),(2,1)\}$
false	false	true	$\{(1,2)\}$
false	false	false	$\{(1,2),(2,3)\}$

1.30. (a) reflexive, not symmetric, transitive.

(b) reflexive (since we are considering a relation on the set of *nonempty* subsets of $\mathcal{N}$), symmetric, not transitive.

(c) not reflexive, symmetric, transitive.

1.31. The relation in (b) would no longer be reflexive.

1.32. not reflexive: $\exists x(\neg xRx)$

not symmetric: $\exists x(\exists y(xRy \wedge \neg yRx))$

not transitive: $\exists x(\exists y(\exists z(xRy \wedge yRz \wedge \neg xRz)))$

1.33(e) It's easy to see that R is reflexive and symmetric. If xRy and yRz, then there are numbers j and k so that $x_i = y_i$ for every $i \geq j$ and $y_i = z_i$ for every $i \geq k$. Therefore, if m is the larger of the two numbers j and k, $x_i = z_i$ for every $i \geq m$.

1.34. We observe first that two elements of A (i.e., two subsets of S) are related if and only if they are the same size. (Any two-element set is related to any other two-element set, and

it is not related to any three-element set.) An equivalence class must contain precisely all k-element subsets of S, for some k with $0 \leq k \leq 10$. There are eleven equivalence classes, one for each value of k. The equivalence class containing $\{a, b\}$ is the set of all two-element subsets of S, of which there are 45.

1.35. $\{0\}$ is an equivalence class, and for each nonzero real number x, $\{x, -x\}$ is an equivalence class.

1.36. (a) n (b) m

1.37. Two statements involving the propositional variables p, q, and r are equivalent if and only if they have the same truth table—in other words, they have the same truth values for each of the eight possible ways of assigning truth values to p, q, and r. Since a truth table has eight rows, there are 2^8 distinct truth tables, and since distinct truth tables correspond to distinct equivalence classes, there are $2^8 = 256$ equivalence classes.

1.38. $L^+ = L^*$ if and only if $\Lambda \in L$.

1.39. (a) $\{a\}^*$ and $\{x \in \{a,b\}^* \mid |x| \text{ is even}\}$ are two examples.
(b) $\{a\}^+$ and $\{x \in \{a,b\}^* \mid |x| \text{ is odd}\}$ are two examples..

1.40. (a) $L_1 = \{a\}$, $L_2 = \{aa\}$.
(b) $L_1 = \{a\}$, $L_2 = \{\Lambda, a\}$.

1.41. (a) $L_1^* \cup L_2^*$ is always a subset of $(L_1 \cup L_2)^*$. This is true because $L_1 \subseteq L_1 \cup L_2$, so that $L_1^* \subseteq (L_1 \cup L_2)^*$, and similarly $L_2^* \subseteq (L_1 \cup L_2)^*$. $L_1 = \{a\}$, $L_2 = \{b\}$ is an example where the opposite inclusion does not hold. For example, $ab \in (L_1 \cup L_2)^*$ and $ab \notin L_1^* \cup L_2^*$.
(b) $L_1 = \{aa, aaaaa\}$, $L_2 = \{aaa, aaaaa\}$. Then L_1^* contains a^i for every i except $i = 1$ and $i = 3$, and L_2^* contains a_i for every i except $i = 1$, $i = 2$, $i = 4$, and $i = 7$. Therefore, $L_1^* \cup L_2^* = (L_1 \cup L_2)^* = \{\Lambda\} \cup \{a^i \mid i \geq 2\}$.

1.43. For any language L, we have $L \subseteq L^+ \subseteq L^*$. From Exercise 1.42, we obtain $L^* \subseteq (L^+)^* \subseteq (L^*)^*$. We also have $L^* \subseteq (L^*)^+ \subseteq (L^*)^*$. Therefore, in order to show that all four languages are equal, it is sufficient to show that $(L^*)^* \subseteq L^*$. But an element of $(L^*)^*$ is formed by concatenating zero or more elements of L^*, each of which is the concatenation of zero or more elements of L. The result is a concatenation of zero or more elements of L.

1.44. Clearly, $|AB| \leq |A||B|$, since an element of AB is specified by first choosing an element of A and then choosing an element of B, and there are $|A||B|$ ways to do this. However, several of these choices might produce the same result. Let $A = B = \{a, aa\}$, for example. $|A||B| = 4$, but $|AB| = |\{aa, aaa, aaaa\}| = 3$.

1.45. One description of $\{a, ab\}^*$ is $\{x \in \{a,b\}^* \mid x \text{ does not start with } b \text{ and does not contain the substring } bb\}$.

1.46. (a) $L = \{a, ba\}^*$

(b) If there were such an S, then b would be an element of S^* (since it does not contain the substring bb), and therefore bb would be (since the concatenation of two elements of S^* is an element of S^*). But this is impossible.

1.47. (a) The two languages are equal.

(b) $(L_1 \cap L_2)^* \subsetneq L_1^* \cap L_2^*$, and they are not always equal. For example, let $L_1 = \{a\}$ and $L_2 = \{aa\}$. Then $L_1^* \cap L_2^*$ contains the string aa, and $(L_1 \cap L_2)^* = \emptyset$.

(c) Neither is necessarily a subset of the other. For example, if $L_1 = \{a\}$ and $L_2 = \{b\}$, $aabb \in L_1^*L_2^* - (L_1L_2)^*$ and $abab \in (L_1L_2)^* - L_1^*L_2^*$.

(d) The two languages are equal.

1.48. One approach is to use the logic underlying Venn diagrams, without actually relying on the diagrams, by considering the eight cases separately. For example, in the case $x \in A \cap B' \cap C$, then $x \in A$ and $x \in B \oplus C$, so $x \notin A \oplus (B \oplus C)$; in addition, $x \in A \oplus B$ and $x \in C$, so $x \notin (A \oplus B) \oplus C$. In the case $x \in A' \cap B' \cap C'$, then $x \notin A$ and $x \notin B \oplus C$, so $x \notin A \oplus (B \oplus C)$; also, $x \notin (A \oplus B)$ and $x \notin C$, so $x \notin (A \oplus B) \oplus C$. The remainder of the proof is to show that in each of the other six cases also, $x \in A \oplus (B \oplus C)$ if and only if $x \in (A \oplus B) \oplus C$.

1.49. (c)

$$\begin{aligned}
|A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| \\
&\quad -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
&\quad +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\
&\quad -|A \cap B \cap C \cap D|
\end{aligned}$$

1.50. (a) 4 (b) One algorithm is described by the following pseudocode, which processes the symbols left-to-right, starting with the leftmost bracket.

```
read ch, the next nonblank character;
e = 0; (element count)
u = 1; (current number of unmatched left brackets)
while there are nonblank characters remaining
{ read ch, the next nonblank character;
  if ch is a left bracket
    u = u + 1;
  else if ch is a right bracket
    u = u - 1;
  if u = 1 and ch is either 0 or a right bracket
    e = e + 1;
}
```

1.51. (a) $\{a\}$ (b) $\{x \mid |x-a|<1\}$

1.52. No. If A and B are distinct and both are nonempty, then either there is an $a \in A-B$ or there is a $b \in B-A$. In the first case, for any $x \in B$, the pair (a, x) is in $A \times B$ but not in $B \times A$; in the second case, for any $x \in A$, the pair (x, b) is in $A \times B$ but not in $B \times A$.

1.53(a) If either A or B is a subset of the other, then the two sets are equal. If not, then $2^A \cup 2^B \subseteq 2^{A \cup B}$; however, if $a \in A-B$ and $b \in B-A$, then the subset $\{a, b\}$ is an element of $2^{A \cup B}$ but is not in $2^A \cup 2^B$.

(b) $2^A \cap 2^B = 2^{A \cap B}$, because a set is a subset of $A \cap B$ if and only if it is both a subset of A and a subset of B.

(c) $2^{A \oplus B}$ can never be a subset of $2^A \oplus 2^B$, because $\emptyset$ is an element of the first but not the second. In the case where $A \cap B = \emptyset$, we have $2^A \oplus 2^B \subseteq 2^{A \oplus B}$. If $A \cap B \neq \emptyset$, however, and $A \neq B$, then a set containing an element of $A \cap B$ as well as an element of $A \oplus B$ is an element of $2^A \oplus 2^B$ that is not in $2^{A \oplus B}$.

(d) Again, the first set can never be a subset of the second, because $\emptyset \in 2^{A'}$ and $\emptyset \notin (2^A)'$. Every element of $2^{A'}$ that is not empty is an element of $(2^A)'$. $(2^A)' \subseteq 2^{A'}$ if and only if either A or A' is empty.

1.54. $(p \wedge q) \vee (\neg p \wedge \neg q)$

1.55. (a) $\exists x(P(x)) \wedge \forall x(\forall y((P(x) \wedge P(y) \rightarrow x=y)$ (The first part of the statement says that there is at least one such x; the second says that there is at most one.) Another possibility is $\exists x(P(x) \wedge \forall y(P(y) \rightarrow y=x))$.

(b) $\exists x(\exists y(P(x) \wedge P(y) \wedge x \neq y))$

1.56. (a) The second logically implies the first, but not vice-versa. For example, let $p(x)$ be the statement "x is even" and $q(x)$ the statement "x is odd". Then the first statement is true and the second is false.

(b) Each logically implies the other.

(c) Each logically implies the other.

(d) The first logically implies the second, but not vice-versa, as illustrated by the statements $p(x)$ and $q(x)$ in part (a).

1.57. (a) Yes. If $y \in f(S \cup T)$, then $y=f(x)$ for some $x \in S \cup T$, and in either case ($x \in S$ or $x \in T$), $y \in f(S) \cup f(T)$.

(b) Yes. Clearly $f(S) \subset f(S \cup T)$ and $f(T) \subset f(S \cup T)$; therefore, the union $f(S) \cup f(T)$ is also a subset of $f(S \cup T)$.

(c) Yes. If $y \in f(S \cap T)$, then $y=f(x)$ for some $x \in S \cap T$. Therefore, $y \in f(S)$, because $x \in S$, and $y \in f(T)$, because $x \in T$.

(d) No. If $y \in f(S) \cap f(T)$, then $y=f(x_1)$ for some $x_1 \in S$, and $y=f(x_2)$ for some $x_2 \in T$. However, x_1 and x_2 may be different, and there is no reason to assume that there is a single $x \in S \cap T$ with $y=f(x)$. For an example, let $A=\{a, b\}$, $B=\{c\}$, $S=\{a\}$, and $T=\{b\}$, and let $f: A \rightarrow B$ be the only possible function from A to B, the one with

$f(a) = f(b) = c$. Then $c \in f(S) \cap f(T)$, but $c \notin f(S \cap T)$ because $S \cap T = \emptyset$.

(e) In part (d), if f is one-to-one, the answer is yes.

1.58. (a) We can define f by saying, for each $S \in 2^X$, that $f(S)$ is the function $F_S : X \to \{0,1\}$ such that $F_S(x) = 1$ if $x \in S$ and $F_S(x) = 0$ if $x \notin S$. f is one-to-one because if $f(S) = F_S = f(T) = F_T$, then the set $\{x \in X \mid F_S(x) = 1\}$, which is just S, is the same as $\{x \in X \mid F_T(x) = 1\}$, which is T. f is onto, because for any function $F : X \to \{0,1\}$, if $S = \{x \in X \mid F(x) = 1\}$, F is just the function $F_S = f(S)$.

(b) We can define g by saying, for each function $t : X \to \{0,1\}$, that $g(t)$ is the n-tuple $(t(1), t(2), \ldots, t(n))$. It's easy to see that g is a bijection.

(c) We can define h by saying, for each n-tuple $N = (i_1, i_2, \ldots, i_n) \in \{0,1\}^n$, that $h(N)$ is the subset $\{x \in X \mid i_x = 1\}$. It is easy to see that h is a bijection.

1.59. (a) No, because the two subsets A and $\mathcal{N} - A$ are the same as the two subsets $\mathcal{N} - A$ and A. (That is, a partition of this type is an *unordered* pair of subsets, and if we start with one unordered pair, switching the two subsets produces the same unordered pair.) This means that f is not one-to-one.

(b) Yes. g is one-to-one, because if $g(A) = g(B)$, then the set $\{A, \mathcal{N} - A\}$ is the same as the set $\{B, \mathcal{N} - B\}$. Because A and B are both subsets of E, and $\mathcal{N} - A$ must therefore contain odd integers, A cannot be $\mathcal{N} - B$, and so A and B must be equal.

1.60. (a) For each $x \in I_0$, $x \in S$ for every $S \in \mathcal{S}$, and so $x \in \cap_{S \in \mathcal{S}} S$. If $x, y \in T$, then $x, y \in S$ for every $S \in \mathcal{S}$; therefore, since each $S \in \mathcal{S}$ is closed under $\circ$, $x \circ y \in S$ for each $S \in \mathcal{S}$; therefore, $x \circ y \in T$.

(b) Suppose $I_0 \subseteq T_1$ and T_1 is closed under $\circ$. Then $T_1 \in \mathcal{S}$, by definition of $\mathcal{S}$. For any $x \in T$, $x \in S$ for every $S \in \mathcal{S}$, and so $x \in T_1$.

1.61. The error is in the statement "choose some $b \in A$ so that aRb"—there may be no such b. The argument does prove that if R is symmetric and transitive, and for any x there is a y with xRy, then R is reflexive.

1.62(a) The same as the number of subsets of $A \times A$, or $2^{(n^2)}$.

(b) Of the n^2 elements in $A \times A$, n are of the form (i, i). Specifying a reflexive relation on A the same as specifying the subset of the remaining $n^2 - n$ pairs that are in the relation in addition to these n. The answer is therefore $2^{n^2 - n}$.

(c) To make the solution easier to describe, take the set A to be $\{1, 2, \ldots, n\}$. If R is symmetric, specifying R is accomplished by specifying the pairs (i, j) in R for which $i \le j$. Therefore, the number of symmetric relations on A is the number of subsets of the set $\{(i,j) \mid 1 \le i \le j \le n\}$. Since this set has $n(n+1)/2$ elements, the number of symmetric relations is $2^{n(n+1)/2}$.

(d) The number of subsets of $\{(i,j) \mid 1 <= i < j <= n\}$, which is $2^{n(n-1)/2}$.

1.63. (a) Another way to describe R^s is $\{(x,y) \mid xRy \text{ or } yRx\}$; now it is easy to see that R^s is symmetric. If R_1 is a symmetric relation with $R \subseteq R_1$, then for every $x, y \in A$, if

xRy, then $(x,y) \in R_1$ (because $R \subseteq R_1$) and $(y,x) \in R_1$ (because R_1 is symmetric); therefore, $R^s \subseteq R_1$.

(b) Suppose xR^ty and yR^tz, and let R_1 be any transitive relation containing R. Then (x,y) and (y,z) are in R_1, since they are in R, and so $(x,z) \in R_1$ because R_1 is transitive. Since this is true for any transitive R_1 containing R, $(x,z) \in R^t$. Furthermore, if R_1 is any transitive relation containing R, then R_1 is one of the relations of which R^t is the intersection, so that $R^t \subseteq R_1$.

(c) R^u and R^t are not always the same. For an example, let $R = \{(n, n+1) \mid n \in \mathcal{N}\}$. Then $R^u = \{(n,m) \mid n \in \mathcal{N} \text{ and } (m = n+1 \text{ or } m = n+2)\}$, whereas R^t contains every pair $(n, n+k)$ for which $k > 0$.

1.64. The function $f : 2^S \to \mathcal{N}$ defined by $f(A) = |A|$ (the number of elements of A) is such a function.

1.65. The function f defined by $f(i) = i \bmod n$ is such a function. (Two integers are congruent mod n if and only if they have the same remainder mod n.)

1.66. We may take B to be the set of equivalence classes with respect to the relation R, and f to be the function defined by $f(x) = [x]$ (the equivalence class containing x).

1.67. Suppose first that S is a maximal pairwise inequivalent set. Let C be any equivalence class. Since S is pairwise inequivalent, S cannot contain more than one element of C, and since S is maximal S must contain at least one (if it didn't, we could add one without destroying the pairwise inequivalent property). On the other hand, if S is any set containing exactly one element from each equivalence class, then S is pairwise inequivalent, because an element of one equivalence class cannot be related to an element from any other equivalence class; and S is maximal, because for any $x \in A$, S contains an element of the equivalence class $[x]$.

1.68. Suppose that $R_1 \subset R_2$. Consider a subset S in the partition P_2 and any element $x \in S$. Let $[x]_1$ be the equivalence class containing x with respect to R_1. ($[x]_1$ is the subset in the partition P_1 containing x.) Then for any $y \in [x]_1$, since xR_1y and $R_1 \subseteq R_2$, $y \in S$. In other words, $[x]_1 \subseteq S$. It follows that S is the union of all the sets $[x]_1$, for the elements $x \in S$. On the other hand, if every subset S in the partition P_2 is the union of subsets in the partition P_1, then for any x and y with xR_1y, x and y are in the same equivalence class relative to R_1, so they are in the same subset in the partition P_2, so xR_2y.

1.69. Obviously, $xy = yx$ if for some string α, $x = \alpha^j$ and $y = \alpha^k$. In fact, this is the only way xy and yx can be equal, although the proof is a little complicated.

Suppose that $xy = yx$. If we can find a string α so that $x = \alpha^j$ and $y = \alpha^k$ for some k and some j, then obviously $|\alpha|$ is a divisor of both $|x|$ and $|y|$. With this in mind, let d be the greatest common divisor of $|x|$ and $|y|$. Then we can write $x = x_1x_2 \ldots x_p$ and $y = y_1y_2 \ldots y_q$, where all the x_i's and y_i's are of length d and p and q have no common factors. (What we'd like to show is that these strings are all identical.)

Since $xy = yx$, it must be true that $x^q y^p = y^p x^q$, because starting with one of these strings, we can obtain the other by repeated transpositions of x and y. $x^q y^p$ and $y^p x^q$ are both strings of length $2pqd$, and the prefixes of length pqd are x^q and y^p, respectively; therefore, these two strings are equal. Now $x^q = (x_1 x_2 \dots x_p)^q$. This means that in the string x^q, the substring x_1 occurs starting at positions 1, $pd + 1$, $2pd + 1$, $\dots (q-1)pd + 1$. In the string y^p, the substring of length d starting at position $ipd + 1$ is y_{r_i}, where $r_i = ip(\text{mod } q) + 1$. Since p and q have no common factors, it is easy to check that all the numbers $r_0, r_1, \dots, r_{q-1}$ are distinct. This means, however, that all the strings $y_1, y_2, \dots, y_q$ are the same, say α, and this makes it clear that all the x_i's are equal to α as well.

1.70. If there were such an L, L^* would have to contain both the strings ab and aa, since these are both elements of $\{aa, bb\}^*\{ab, ba\}^*$. However, L^* would then contain $abaa$, which is not an element of $\{aa, bb\}^*\{ab, ba\}^*$.

1.71. No. Suppose $L = L_1 L_2$ and neither L_1 nor L_2 is $\{\Lambda\}$. Since any even-length string of 0's is in L, there are arbitrarily long strings of 0's that must be in either L_1 or L_2. Similarly, there are arbitrarily long strings of 1's that are in L_1 or L_2. It is not possible for L_1 to have a nonnull string of 0's and L_2 to have a nonnull string of 1's, since the concatenation could not be in L; similarly, there can't be a string of 1's in L_1 and a string of 0's in L_2. The only possibilities, therefore, are for all the strings of 0's and all the strings of 1's to be in L_1 or for all these strings to be in L_2. Suppose, however, that they are all in L_1. Let y be a nonnull string in L_2. Then y contains both 0's and 1's. L_1 contains a string x of 0's with $|x| \geq |y|$. But then xy, which is in $L_1 L_2$, has 1's in the second half and not in the first, so that it can't be in L. A similar argument shows that L_2 can't contain all the strings of 0's and the strings of 1's. This argument shows that our original assumption, that $L = L_1 L_2$ and neither L_1 nor L_2 is $\{\Lambda\}$, must not be true.