Chapter 2

Mathematical Induction and Recursive Definitions

2.5. The proof is identical except that the proof of the induction step now goes like this. Since $|x| = k + 1$ and $x = 0y1$, $|0y| = k$. If y ends with 0, then x ends with the substring 01. If y ends with 1, then $0y$ begins with 0 and ends with 1; by the induction hypothesis, $0y$ contains the substring 01, and therefore x does also.

2.6. To show that for any $n \geq 1$, if x contains n 0's and $x = 0y1$, then x contains the substring 01.
Basis step. If x contains only one 0 and $x = 0y1$, then x begins with 01.
Induction hypothesis. $k \geq 1$, and if x contains k 0's and $x = 0y1$, then x contains the substring 01.
Statement to be shown in induction step. If x contains $k + 1$ 0's and $x = 0y1$, then x contains the substring 01.
Proof. Consider the string $y1$, which has k 0's. If y starts with 1, then x begins with 01. Otherwise, $y1$ starts with 0 and ends with 1. Therefore, by the induction hypothesis, $y1$ contains the substring 01, and so x does also.

2.7. Proving the basis step would be as difficult as proving the original statement.

2.8. **Induction hypothesis.** $k \geq 0$ and $\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$.
Statement to be shown in induction step. $\sum_{i=1}^{k+1} i^2 = (k+1)((k+1)+1)(2(k+1)+1)/6$
Proof of induction step.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= k(k+1)(2k+1)/6 + (k+1)^2 \\
&= (k+1)(k(2k+1)/6 + (k+1)) \\
&= (k+1)(2k^2 + 7k + 6)/6 \\
&= (k+1)(k+2)(2k+3)/6 \\
&= (k+1)((k+1)+1)(2(k+1)+1)/6
\end{aligned}$$

(The second equality is because of the induction hypothesis.)

2.9. **Proof of induction step** (using ordinary induction).

$$\begin{aligned}
\sum_{i=1}^{k+1} (a_i - a_{i-1}) &= \sum_{i=1}^{k} (a_i - a_{i-1}) + (a_{k+1} - a_k) \\
&= (a_k - a_0) + (a_{k+1} - a_k) = a_{k+1} - a_0
\end{aligned}$$

2.10. **Proof of induction step** (using ordinary induction).

$$\begin{aligned}7+13+19+\ \ldots\ &+(6k+1)+(6(k+1)+1) = (7+13+\ldots+6k+1)+(6(k+1)+1)\\ &= k(3k+4)+(6k+7) \text{ (using the induction hypothesis)}\\ &= 3k^2+10k+7 = (k+1)(3k+7) = (k+1)(3(k+1)+4)\end{aligned}$$

2.11. **Proof of induction step** (using ordinary induction).

$$\begin{aligned}\sum_{i=1}^{k+1}\frac{1}{i(i+1)} &= \sum_{i=1}^{k}\frac{1}{i(i+1)}+\frac{1}{(k+1)(k+1+1)}\\ &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\ &= \frac{k(k+2)+1}{(k+1)(k+2)}\\ &= \frac{(k+1)^2}{(k+1)(k+2)}\\ &= \frac{k+1}{k+2}\end{aligned}$$

2.12. (a)

$$\begin{aligned}C(n-1,i-1)+C(n-1,i) &= \frac{(n-1)!}{(i-1)!(n-1-(i-1))!}+\frac{(n-1)!}{i!(n-1-i)!}\\ &= \frac{(n-1)!}{(i-1)!(n-i)!}+\frac{(n-1)!}{i!(n-1-i)!}\\ &= \frac{(n-1)!i}{i(i-1)!(n-i)!}+\frac{(n-1)!(n-i)}{i!(n-i)(n-i-1)!}\\ &= \frac{(n-1)!i}{i!(n-i)!}+\frac{(n-1)!(n-i)}{i!(n-i)!}\\ &= \frac{(n-1)!(i+n-i)}{i!(n-i)!}\\ &= \frac{n(n-1)!}{i!(n-i)!}\\ &= \frac{n!}{i!(n-i)!} = C(n,i)\end{aligned}$$

(b) **Proof of induction step** (using ordinary induction).

$$\sum_{i=0}^{k+1}C(k+1,i) = C(k+1,0)+\sum_{i=1}^{k}C(k+1,i)+C(k+1,k+1)$$

$$\begin{aligned}
&= 1+\sum_{i=1}^{k}(C(k,i-1)+C(k,i))+1 \\
&= 1+\sum_{i=0}^{k-1}C(k,i)+\sum_{i=1}^{k}C(k,i)+1 \\
&= \sum_{i=0}^{k}C(k,i)+\sum_{i=0}^{k}C(k,i) \\
&= 2\sum_{i=0}^{k}C(k,i) \\
&= 2(2^k) = 2^{k+1}
\end{aligned}$$

2.13. **Proof of induction step** (using ordinary induction).

$$\begin{aligned}
\sum_{i=0}^{k+1} r^i &= \sum_{i=0}^{k} r^i + r^{k+1} \\
&= \frac{1-r^{k+1}}{1-r} + r^{k+1} \\
&= \frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r} \\
&= \frac{1-r^{(k+1)+1}}{1-r}
\end{aligned}$$

2.14. **Proof of induction step** (using ordinary induction).

$$\begin{aligned}
1+\sum_{i=1}^{k+1} i*i! &= 1+\sum_{i=1}^{k} i*i! + (k+1)*(k+1)! \\
&= (k+1)! + (k+1)(k+1)! \\
&= (k+1)!(1+k+1) = (k+2)!
\end{aligned}$$

2.15. **Proof of induction step** (using ordinary induction).

$$(k+1)! = (k+1)*k! \geq 2*k! > 2*2^k = 2^{k+1}$$

The first inequality uses the fact that $k \geq 1$, and the second inequality uses the induction hypothesis.

2.16. We can consider any $m \geq 0$, and then prove, by induction on n, the statement: for every $n \geq m$, $a_m \leq a_n$.
Statement to be proved in induction step. $a_m \leq a_{k+1}$

Proof of induction step. By the induction hypothesis, $a_m \le a_k$; by the original assumption, $a_k \le a_{k+1}$. Therefore, by the transitivity of the $\le$ relation, $a_m \le a_{k+1}$.

2.17. **Basis step.** The statement to be proved is $(1+x)^0 \ge 1 + 0x$, which is clear.
Induction hypothesis. $k \ge 0$ and $(1+x)^k \ge 1 + kx$.
Statement to be proved in induction step. $(1+x)^{k+1} \ge 1 + (k+1)x$.
Proof.

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k * (1+x)\\ &\ge (1+kx) * (1+x)\\ &= 1 + kx + x + kx^2\\ &= 1 + (k+1)x + kx^2\\ &\ge 1 + (k+1)x\end{aligned}$$

The first inequality uses the induction hypothesis, the fact that $x+1 > 0$, and the algebraic fact that if $A \ge B$ and $C > 0$, then $AC \ge BC$ (where $A = (1+x)^k$, $B = 1 + kx$, and $C = 1 + x$). The second inequality uses the fact that for any x, $x^2 \ge 0$.

2.18. **Basis step.** If we let $k_1 = 2$, then $\sum_{i=1}^{k_1} 1/i = 1/1 + 1/2 > 1$.
Induction hypothesis. $j \ge 1$, and there is an integer k_j so that $\sum_{i=1}^{k_j} 1/i > j$.
Statement to be proved in induction step. There is an integer k_{j+1} so that $\sum_{i=1}^{k_{j+1}} 1/i > j+1$.
Proof. First we write

$$\sum_{i=1}^{k_{j+1}} 1/i = \sum_{i=1}^{k_j} 1/i + 1/(k_j+1) + \ldots + 1/k_{j+1}$$

According to the induction hypothesis, the first sum on the right side is greater than j. Therefore, all we need to show is that we can choose k_{j+1} so that the sum of the remaining terms is at least 1. The hint suggests that we consider the sum $1/(k_j+1) + 1/(k_j+2) + \ldots + 1/(2k_j)$. This is the sum of k_j terms, each of which is at least $1/(2k_j)$; therefore, this sum is greater than $1/2$. Now we may repeat this step, starting with $1/(2k_j+1)$: the sum $1/(2k_j+1) + 1/(2k_j+2) + \ldots + 1/(2(2k_j))$ is also greater than $1/2$. Now we choose $k_{j+1} = 4k_j$, and we have

$$\sum_{i=1}^{k_{j+1}} 1/i = \sum_{i=1}^{k_j} 1/i + \sum_{i=k_j+1}^{2k_j} 1/i + \sum_{i=2k_j+1}^{4k_j} 1/i > j + 1/2 + 1/2 = j+1$$

2.19. **Proof of induction step** (using ordinary induction).

$$\sum_{i=1}^{k+1} i * 2^i = \sum_{i=1}^{k} i * 2^i + (k+1) * 2^{k+1}$$

$$\begin{aligned}
&= (k-1)*2^{k+1} + 2 + (k+1)*2^{k+1} \\
&= (k-1+k+1)*2^{k+1} + 2 \\
&= k*2^{k+2} + 2
\end{aligned}$$

2.20. **Proof of induction step** (using ordinary induction).

$$\begin{aligned}
1+\sum_{i=2}^{k+1}\frac{1}{\sqrt{i}} &= 1+\sum_{i=2}^{k}\frac{1}{\sqrt{i}}+\frac{1}{\sqrt{k+1}} \\
&> \sqrt{k}+\frac{1}{\sqrt{k+1}} \\
&= \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} \\
&> \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} \\
&= \frac{k+1}{\sqrt{k+1}} = \frac{1}{\sqrt{k+1}}
\end{aligned}$$

2.21. **Basis step.** 0 is even, and therefore 0 is either even or odd.
Induction hypothesis. $k \geq 0$, and k is either even or odd.
Statement to be shown in induction step. $k+1$ is either even or odd.
Proof. We consider two cases, depending on whether k is even or odd. If k is even, then there is an integer i with $k = 2i$. In this case $k+1 = 2i+1$. It follows by the definition that $k+1$ is odd, and therefore that $k+1$ is either even or odd. If k is odd, then there is an integer i with $k = 2i+1$. Then $k+1 = 2i+2 = 2(i+1)$. Since there is an integer j (namely, $j = i+1$) for which $k+1 = 2j$, $k+1$ is even, and therefore $k+1$ is either even or odd.

2.22. We formulate the statement as follows: for every $n \geq 1$, if $L^2 \subseteq L$, then $L^n \subseteq L$. (It will then follow that if $L^2 \subseteq L$, then $\cup_{n=1}^{\infty} L^n = L^+ \subseteq L$.)
Basis step. $L^1 = L$, and so $L^1 \subseteq L$. (Here we don't even need the assumption that $L^2 \subseteq L$.
Induction hypothesis. $k \geq 1$, and if $L^2 \subseteq L$, then $L^k \subseteq L$.
Statement to be shown in induction step. If $L^2 \subseteq L$, then $L^{k+1} \subseteq L$.
Proof. If $L^2 \subseteq L$, then $L^k \subseteq L$ because of the induction hypothesis. Therefore, $L^{k+1} = L^k L \subseteq LL$. Therefore, if $LL \subset L$, it follows that $L^{k+1} \subseteq L$.

2.23. We show this statement by structural induction, using the usual recursive definition of Σ^*.
Basis step. To show $f(\Lambda) = \Lambda$. Using the assumption on f, we know that $f(\Lambda) = f(\Lambda\Lambda) = f(\Lambda)f(\Lambda)$. The only string z satisfying the equation $z = zz$ is the string Λ.
Induction hypothesis. x is a string for which $f(x) = x$.
Statement to be shown in induction step. For any $a \in \Sigma$, $f(xa) = xa$.

Proof. For any $a \in \Sigma$, $f(xa) = f(x)f(a)$ by assumption. By the induction hypothesis, $f(x) = x$, and by the assumption on f, $f(a) = a$. Therefore, $f(xa) = xa$.

2.24. **Proof of induction step** (using ordinary induction).
$(k+1)((k+1)^2+5) = (k+1)^3+5(k+1)+5 = k^3+3k^2+3k+1+5(k+1) = k^3+3k^2+8k+6$. This can be rewritten as $k^3 + 5k + (3k^2 + 3k + 6)$, or $k(k^2 + 5) + (3k(k + 1) + 6)$. The induction hypothesis tells us that the first term is divisible by 6. For any k, either k or $k + 1$ is even, so that $k(k + 1)$ is even, and therefore $3k(k + 1)$ is divisible by 6. Therefore, the entire expression is.

2.25. **Basis step.** To show that for every a and b with $0 \leq a < b$, $b^1 - a^1$ is divisible by $b - a$. This is obvious.
Induction hypothesis. $k \geq 1$, and for every a and b with $0 \leq a < b$, $b^k - a^k$ is divisible by $b - a$.
Statement to be shown in induction step. For every a and b with $0 \leq a < b$, $b^{k+1} - a^{k+1}$ is divisible by $b - a$.
Proof. We can write

$$b^{k+1} - a^{k+1} = b * b^k - a * a^k = b * b^k - a * b^k + a * b^k - a * a^k = (b - a) * b^k + a * (b^k - a^k)$$

The first term is obviously divisible by $b - a$, and the induction hypothesis says that the second is also. Therefore, the sum is divisible by $b - a$.

2.26. **Basis step.** $1 = 2^0 * 1$, which is a power of 2 times an odd integer.
Induction hypothesis. $k \geq 1$, and any integer n with $1 \leq n \leq k$ is the product of a power of 2 and an odd integer.
Statement to be shown in induction step. $k + 1$ is the product of a power of 2 and an odd integer.
Proof. We consider two cases. If $k + 1$ is odd, then $k + 1$ is the product of a power of 2 (namely, 2^0) and an odd integer (namely, $k + 1$). If $k + 1$ is even, then $k + 1 = 2 * j$ for some $j \geq 1$. By the induction hypothesis, $j = 2^p * m$ for some nonnegative integer p and some odd integer m. Therefore, $k + 1 = 2 * j = 2^{p+1}m$.

2.27. **Proof of induction step** (using ordinary induction).

$$(\bigcap_{i=1}^{k+1} A_i)' = (\bigcap_{i=1}^{k} A_i \cap A_{k+1})' = (\bigcap_{i=1}^{k} A_i)' \cup A'_{k+1} = \bigcup_{i=1}^{k} A'_i \cup A'_{k+1} = \bigcup_{i=1}^{k+1} A'_i$$

The second equality follows from De Morgan's law, formula (1.12), and the third follows from the induction hypothesis.

2.28. Basis step. There are two subsets of $\{1\}$, namely, $\emptyset$ and $\{1\}$.
Induction hypothesis. $k \geq 1$, and there are 2^k subsets of $\{1, 2, \ldots, k\}$.
Statement to be shown in induction step. There are 2^{k+1} subsets of $\{1, 2, \ldots, k+1\}$.
Proof. By the induction hypothesis, there are 2^k subsets of $\{1, 2, \ldots, k\}$. Each such subset

A determines exactly two subsets of $\{1, 2, \dots, k+1\}$ (namely, A and $A \cup \{k+1\}$), and all of the 2^{k+1} subsets obtained this way are distinct. Furthermore, every subset B of $\{1, 2, \dots, k+1\}$ arises this way, since if $k+1 \notin B$ then B is a subset of $\{1, 2, \dots, k\}$ and otherwise $B = (B - \{k+1\}) \cup \{k+1\}$. Therefore, the number of subsets of $\{1, 2, \dots, k+1\}$ is 2^{k+1}.

2.29. This is a generalization of 2.28, since a subset of $\{1, 2, \dots, n\}$ can be viewed as a function from $\{1, 2, \dots, n\}$ to a two-element set, say {true, false}. Of the two integers m and n, n is the more appropriate one on which to base the induction. A function from $\{1, 2, \dots, k+1\}$ to $\{1, 2, \dots, m\}$ is specified by first saying how it is defined on the elements of $\{1, 2, \dots, k\}$ and then saying how it is defined at $k+1$. By the induction hypothesis, there are m^k ways to do the first, and there are m ways to do the second, since the value can be any one of the m possible values. Therefore, there are $m^k * m = m^{k+1}$ ways to specify the function.

2.30. **Proof of the induction step** (using ordinary induction).

$$\begin{aligned}
\frac{d}{dx}(x^{k+1}) &= \frac{d}{dx}(x^k * x) \\
&= x^k * \frac{d}{dx}x + \frac{d}{dx}(x^k) * x \\
&= x^k * 1 + kx^{k-1} * x = x^k + kx^k = (k+1)x^k
\end{aligned}$$

2.31 **Basis step.** $a_0 = -2 = 2 * 1 - 4 * 1 = 2 * 3^0 - 4 * 2^0$.
Induction hypothesis. $k \geq 0$, and for every n with $0 \leq n \leq k$, $a_n = 2 * 3^n - 4 * 2^n$.
Statement to be shown in induction step. $a_{k+1} = 2 * 3^{k+1} - 4 * 2^{k+1}$.
Proof. If $k = 0$, $a_{k+1} = -2$, which is $2 * 3^1 - 4 * 2^1$. Otherwise,

$$\begin{aligned}
a_{k+1} &= 5a_k - 6a_{k-1} \\
&= 5(2 * 3^k - 4 * 2^k) - 6(2 * 3^{k-1} - 4 * 2^{k-1}) \\
&= 10 * 3^k - 12 * 3^{k-1} - 20 * 2^k + 24 * 2^{k-1} \\
&= 10 * 3^k - 4 * 3^k - 20 * 2^k + 12 * 2^k \\
&= 6 * 3^k - 8 * 2^k = 2 * 3^{k+1} - 4 * 2^{k+1}
\end{aligned}$$

2.32. (a) **Induction Hypothesis.** $k \geq 0$, and for every n with $0 \leq n \leq k$, $f_n < C(13/8)^n$.
Statement to be shown in induction step. $f_{k+1} < C(13/8)^{k+1}$.
Proof. If $k+1 = 1$, then $f_{k+1} = 1 < C(13/8)$, because of the assumption that $C < 8/13$. If $k+1 > 1$, then by definition $f_{k+1} = f_k + f_{k-1}$. According to the induction hypothesis, $f_k < C(13/8)^k$ and $f_{k-1} < C(13/8)^{k-1}$. Therefore,

$$f_{k+1} < C((13/8)^k + (13/8)^{k-1}) = C(13/8)^{k-1}(13/8+1) = C(13/8)^{k-1}(168/64)$$
$$< C(13/8)^{k-1}(169/64) = C(13/8)^{k+1}$$

(b) **Proof of induction step** (using ordinary induction).

$$\begin{aligned}\sum_{i=0}^{k+1} f_i^2 &= \sum_{i=0}^{k} f_i^2 + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 = f_{k+1}(f_k + f_{k+1}) = f_{k+1}f_{k+2}\end{aligned}$$

(c) **Proof of induction step** (using ordinary induction).

$$\begin{aligned}\sum_{i=0}^{k+1} f_i &= \sum_{i=0}^{k} f_i + f_{k+1} \\ &= f_{k+2} - 1 + f_{k+1} = f_{k+3} - 1\end{aligned}$$

2.33. (a) **Proof of induction step** (using ordinary induction).

$$f(k+1) = \sqrt{1+f(k)} < \sqrt{1+2} < \sqrt{2+2} = 2$$

(The first inequality follows from the induction hypothesis, and the fact that if $0 \le a < b$, then $\sqrt{a} < \sqrt{(b)}$.)

(b) **Proof of induction step** (using ordinary induction).

$$f(k+1) = \sqrt{1+f(k)} < \sqrt{1+f(k+1)} = f(k+2)$$

2.34. Assuming that our recursive definition of Σ^* is still the first one in Example 2.15, it would appear that induction on x doesn't work, because in the induction step, we have the expression $|(xa)y|$, which we have no way to simplify. If we used a different recursive definition of Σ^*, in which symbols are added at the left end, and the corresponding recursive definition of $|x|$ (i.e., using the formula $|ax| = |x| + 1$), then it would be possible.

2.35. In the induction step of a proof using structural induction on x, we would have $|(xa)^r| = |ax^r|$. Using Example 2.24, this is $|a| + |x^r|$, which is $|a| + |x|$ by the induction hypothesis. So to complete the proof we only need to show that $|a| = 1$, which is easy: $|a| = |\Lambda a| = |\Lambda| + 1 = 1$.

2.36. We use structural induction, based on the definition of AE. **Basis step.** Every prefix of i contains at least as many ('s as)'s. This is obvious.

Induction hypothesis. For some strings x and y in AE, every prefix of x or y contains

at least as many ('s as)'s.
Statement to be shown in induction step. Every prefix of $(x+y)$ contains at least as many ('s as)'s, and similarly for $(x-y)$.
Proof. We show the first statement, and the proof of the second is virtually identical. Let $z=(x+y)$. A prefix of z must be either Λ, or $(x_1$ for some prefix x_1 of x, or $(x+y_1$, for some prefix y_1 of y, or z. The first of these four cases is obvious. In the second case, since by the induction hypothesis x_1 has at least as many ('s as)'s, the string $(x_1$ actually has *more* left parentheses than right. In the third case, the induction hypothesis implies that both x and y_1 have at least as many ('s as)'s (note: x is a prefix of itself), and so $(x+y_1$ does also. Finally, in the fourth case the induction hypothesis implies that x and y both have at least as many ('s as)'s, from which it follows that $(x+y)$ does also.

2.37. (a) The basis step is easy, since i is in both languages by definition.
Induction Hypothesis. x and y are in GAE.
Statement to be shown in induction step. $(x+y)$ and $(x-y)$ are in GAE.
Proof. The strings $x+y$ and $x-y$ are in GAE, by part (ii) of the definition. Therefore, both $(x+y)$ and $(x-y)$ are also, by part (iii).

The proof of (b) is similar to that in Exercise 2.36. In (c), the number $N(x)$ is the maximum depth of parenthesis nesting in the expression x—i.e., if $N(x)=k$, there are k nested pairs of parentheses in the expression, and nowhere in x are there more than k.

2.38. Suppose we know that for any $k \geq n_0$, if $P(n)$ is true for every n with $n_0 \leq n < k$, then $P(k)$ is true. We can show that $P(n)$ is true for every $n \geq n_0$ using the strong principle of induction. The basis step is to show that $P(n_0)$ is true. However, our assumption for the value $k=n_0$ is that if $P(n)$ is true for every n with $n_0 \leq n < n_0$, then $P(n_0)$ is true, and $P(n)$ *is* true for every n in this range, because there are no n's in this range.

The induction step is to show that if $k \geq n_0$ and $P(n)$ is true for every n with $n_0 \leq n \leq k$, then $P(k+1)$ is true. This follows from the assumption, using $k+1$ instead of k: we know that if $P(n)$ is true for every n with $n_0 \leq n < k+1$, then $P(k+1)$ is true.

2.39. (a) The set of all strings beginning with a.

(b) The set of all strings containing exactly one a.

(c) The set of all strings containing at least one a and not containing the substring ba. It can also be described as the set of all strings containing at least one a in which all the a's precede all the b's.

(d) The set of all strings containing at least one a.

(e) The set of all strings containing at least one a.

(f) The set of all strings starting with a that do not contain the substring aa.

2.40. (Each of the definitions is assumed to contain a final statement that nothing else is in the language.)
(a) $0 \in \mathcal{N}$; for every $n \in \mathcal{N}$, $n+1 \in \mathcal{N}$.
(b) $0 \in S$; for every $n \in S$, both $n-7$ and $n+7$ are in S.
(c) $2 \in T$; $7 \in T$; for every $n \in T$ and every positive integer i, $in \in T$.

(d) $00 \in U$; for every $x \in U$, all the strings $0x$, $1x$, $x0$, and $x1$ are in U.
(e) $\Lambda \in V$; for every $x \in V$, both $0x1$ and $00x1$ are in V.
(f) $\Lambda \in W$; for every $x \in W$, both $0x$ and $00x1$ are in W.

2.41. **Proof of the induction step** (using strong induction).
Suppose $x \in L$ and $|x| = k+1$. Then $x \neq \Lambda$, and so there are two possibilities for x: either $x = 0y$ for some $y \in L$, or $x = 0y1$ for some $y \in L$. In either case, $y = 0^i1^j$ for some i and j with $i \geq j \geq 0$, and since $|y| \leq k$, $y \in A$ by the induction hypothesis. In the first case, $x = 0^{i+1}1^j$ and $i+1 \geq j \geq 0$, so that $x \in A$; in the second case, $x = 0^{i+1}1^{j+1}$ and $i+1 \geq j+1 \geq 0$, so that $x \in A$.

2.42 (b) First, to show $L_2 \subseteq L$. This is an easy proof using structural induction.
Basis step. Both the strings Λ and a are clearly in L, because neither contains the substring aa.
Induction hypothesis. x is a string in L_2 that is an element of L.
Statement to be proved in induction step. bx and abx are in L.
Proof. These are both obvious. If x does not contain the substring aa, neither does bx or abx.

Now to show $L \subseteq L_2$, using strong induction on the length of the string.
Basis step. If $x \in L$ and $|x| = 0$, then $x = \Lambda$, which is in L_2 according to the definition of L_2.
Induction hypothesis. $k \geq 0$, and for any $x \in L$ with $|x| \leq k$, $x \in L_2$.
Statement to be proved in induction step. If $x \in L$ and $|x| = k+1$, then $x \in L_2$.
Proof. If $x = by$, then since x does not contain the substring aa, neither does y. Therefore, $y \in L$, and so by the induction hypothesis, $y \in L_2$. Therefore, according to the first portion of the recursive part of the definition of L_2, $by \in L_2$. Otherwise $x = ay$ for some y (because $x \neq \Lambda$), and since we know that $x \in L$, y cannot start with a. If $y = \Lambda$, then $x = a$, and $x \in L_2$ by the definition of L_2. If $y \neq \Lambda$, then $y = bz$, so that $x = abz$. The string z cannot contain the substring aa, and so $z \in L_2$ by the induction hypothesis. Therefore, by the second portion of the recursive part of the definition of L_2, $abz \in L_2$.

2.43.(c) The strings a_0 and b_0 are clearly palindromes. For $n > 0$,

$$a_{2n} = a_{2n-2}b_{2n-2}b_{2n-2}a_{2n-2}$$

This makes it easy to prove by induction that $a_{2n}^r = a_{2n}$, and so a_{2n} is a palindrome (see Exercise 2.60). A similar argument holds for b_{2n}.

(d) It is easy to see that for every $n \geq 2$, a_n starts with 01 and ends with 10, and b_n starts with 10 and ends with 01. Using this fact, it is easy to show by induction that neither a_n nor b_n contains 000 or 111.

2.44. **Proof of induction step** (using ordinary induction). Suppose $f : A \to B$, A has $k+2$ elements, and B has $k+1$. Let a be any specific element of A. If there is some other

element $a_1 \in A$ with $f(a) = f(a_1)$, then obviously f is not one-to-one. If not, then we may consider the function $f_a : A - \{a\} \to B - \{f(a)\}$ defined by the formula $f_a(x) = f(x)$. By the induction hypothesis, f_a is not 1-1, and therefore f is not.

2.45. The first incorrect statement is "Now observe that C_{k-1} is an element of both T and R." If $k = 1$, there is no element called C_{k-1}. (The only thing wrong with the proof, in other words, is that it doesn't work when $k = 1$.)

2.46. Suppose p and q are distinct primes and n is divisible by both p and q. Then $n = pm$ for some m. Therefore, pm is divisible by q. Therefore, by the generally accepted fact, either p is divisible by q or m is divisible by q. However, p cannot be, because p, being prime, is divisible only by itself and 1, and q is not either of these. Therefore, m is divisible by q, and so pm is also.

2.47. This follows almost immediately from the previous exercise.

2.48. (The idea of the proof follows that of the proof in Example 2.3, except that we have to be a little more careful.) Suppose for the sake of contradiction that $\sqrt{n} = a/b$ for positive integers a and b. Then $b^2n = a^2$. We know n is not a perfect square; for reasons that will be clear in a minute, we want the number n not to be divisible by any perfect square bigger than 1. Of course, it may be, so let c be the biggest integer whose square divides n, and let $n = c^2m$. Then $b^2c^2m = a^2$. Rewrite this as $d^2m = a^2$. (This is the same as what we started with—i.e., $\sqrt{m} = a/d$—except we have a little more information about m.) Just as in the proof in Example 2.3, we can divide out factors common to the integers a and d, so that we may assume they have no common factors.

Now let p be any prime factor of m, so that $m = pm'$. Then p is a factor of a^2. Using the generally accepted fact in Exercise 2.46, p is a factor of a. Let $a = ep$. Then $a^2 = e^2p^2 = d^2m = d^2pm'$. Canceling p, we get $e^2p = d^2m'$. We know that p is not a factor of m', because $m = pm'$ is not divisible by p^2. Therefore, using the generally accepted fact again, p is a factor of d^2. Using it once more, p is a factor of d. Now we have a contradiction, however, because p is a factor of both a and d, and we assumed a and d had no common factors.

2.49. The induction step (using ordinary induction) follows from the inequality

$$\frac{2k\sqrt{k}}{3} + \sqrt{k+1} > \frac{2(k+1)\sqrt{k+1}}{3}$$

By subtracting $\frac{2\sqrt{k+1}}{3}$ from both sides and multiplying both sides by 3, we see that this is equivalent to

$$2k\sqrt{k} + \sqrt{k+1} > 2k\sqrt{k+1}$$

and the truth of this inequality can be seen easily by squaring both sides.

2.50. **Basis step.** Since $18 = 1 * 4 + 2 * 7$, we may choose $i_{18} = 1$ and $j_{18} = 2$.

Induction hypothesis. $k \geq 18$, and there exist integers i_k and j_k so that $k = i_k * 4 + j_k * 7$.
Statement to be shown in induction step. There exist integers i_{k+1} and j_{k+1} so that $k + 1 = i_{k+1} * 4 + j_{k+1} * 7$.
Proof. We consider first the case in which $j_k \geq 1$. In this case we may write $k + 1 = i_k * 4 + j_k * 7 + 1 = (i_k + 2) * 4 + (j_k - 1) * 7$, and we may therefore take $i_{k+1} = i_k + 2$ and $j_{k+1} = j_k - 1$. Otherwise, $k = i_k * 4$. We know $k \geq 18$; if $k \geq 20$, then $i_k \geq 5$, and this allows us to take $i_{k+1} = i_k - 5$ and $j_{k+1} = 3$, since $i_k * 4 + 1 = (i_k - 5) * 4 + 3 * 7$. We may therefore complete the proof by considering $k = 18$ and $k = 19$ separately. $18 + 1 = 3 * 4 + 1 * 7$, and $19 + 1 = 5 * 4 + 0 * 7$.

2.51. **Proof of induction step** (using ordinary induction). By the induction hypothesis, the number of subsets of $\{1, 2, \ldots, k\}$ having an even number of elements is 2^{k-1}. Since the total number of subsets is 2^k, the number having an odd number of elements is also 2^{k-1}. If $A \subseteq \{1, 2, \ldots, k+1\}$ and $|A|$ is even, then either $A = A_1 \cup \{k + 1\}$, where A_1 is a subset of $\{1, 2, \ldots, k\}$ having an odd number of elements, or A is itself a subset of $\{1, 2, \ldots, k\}$ having an even number of elements. There are 2^{k-1} subsets of the first type and 2^{k-1} of the second type, and so there are 2^k altogether.

2.52. Suppose that the three given statements are true. For each $n \geq N$, let us consider the statement

$$Q(n) = P(N) \wedge P(N + 1) \wedge P(N + 2) \wedge \ldots \wedge P(n)$$

and try to show that $Q(n)$ is true for every $n \geq N$, using the ordinary principle of induction. It will then follow that $P(n)$ is true for every $n \geq N$.

The statement in the basis step is one of the three statements we are assuming is true. The induction hypothesis is the statement

$$k \geq N \text{ and } P(N) \wedge P(N + 1) \wedge \ldots \wedge P(k) \text{ is true.}$$

We wish to show that

$$P(N) \wedge P(N + 1) \wedge \ldots \wedge P(k + 1) \text{ is true.}$$

But the statement that the first formula implies the second is also one of the three statements we are assuming.

2.53. (a) We use the strong principle of induction. In the induction step, we consider the case $k + 1 = 1$ separately. If $k + 1 > 1$, $f_{k+1} = f_k + f_{k-1} = c(a^k - b^k + a^{k-1} - b^{k-1}) = c(a^{k-1}(a + 1) - b^{k-1}(b + 1))$. It is easy to verify that $a + 1 = a^2$ and $b + 1 = b^2$, and this implies the result.

(b) In the induction step, we have

$$\begin{aligned} f_{k+2}^2 &= f_{k+1} f_{k+2} + f_k f_{k+2} \\ &= f_{k+1}(f_k + f_{k+1}) + f_k f_{k+2} \\ &= f_{k+1}^2 + f_k f_{k+2} + f_k f_{k+1} \end{aligned}$$

$$\begin{aligned}
&= f_{k+1}^2 + (f_{k+1}^2 - (-1)^{k+1}) + f_k f_{k+1} \\
&= 2f_{k+1}^2 - (-1)^{k+1} + f_k f_{k+1} \\
&= f_{k+1}(2f_{k+1} + f_k) + (-1)^{k+2} \\
&= f_{k+1}(f_{k+1} + (f_k + f_{k+1})) + (-1)^{k+2} \\
&= f_{k+1}(f_{k+1} + f_{k+2}) + (-1)^{k+2} \\
&= f_{k+1}f_{k+3} + (-1)^{k+2}
\end{aligned}$$

The fourth equality uses the induction hypothesis.

2.54. We first show that for any $n \geq 1$, n can be expressed as a sum of distinct powers of 2. This is true for $n = 1$, since $1 = 2^0$.
Induction hypothesis. $k \geq 1$, and any positive integer $\leq k$ is the sum of distinct powers of 2.
Statement to be shown in induction step. $k + 1$ is the sum of distinct powers of 2.
Proof. We know that $2^1 \leq k + 1$. Let i be the largest integer such that $2^i \leq k + 1$. If $2^i = k + 1$, then we have the result we want. Otherwise, the induction hypothesis applied to the positive integer $j = k + 1 - 2^i$ tells us that j is the sum of distinct powers of 2. Since $j < 2^i$ (because otherwise 2^{i+1} would be $\leq k + 1$), none of these powers is as large as i, and therefore, $k + 1 = j + 2^i$ is the sum of distinct powers of 2.

For the second part, rather than using induction on n, we show that for any $m \geq 0$, if $\sum_{i=0}^{m} a_i * 2^i = \sum_{i=0}^{m} b^i * 2^i$ and each a_i is 0 or 1 and each b_i is 0 or 1, then $a_i = b_i$ for each i. The basis step is straightforward.
Induction hypothesis. $k \geq 0$, and if $\sum_{i=0}^{k} a_i * 2^i = \sum_{i=0}^{k} b^i * 2^i$ and each a_i is 0 or 1 and each b_i is 0 or 1, then $a_i = b_i$ for each i.
Statement to be shown in induction step. if $\sum_{i=0}^{k+1} a_i * 2^i = \sum_{i=0}^{k+1} b^i * 2^i$ and each a_i is 0 or 1 and each b_i is 0 or 1, then $a_i = b_i$ for each i.
Proof. We first show that $a_{k+1} = b_{k+1}$. If this is not true, assume for the sake of concreteness that $a_{k+1} = 0$ and $b_{k+1} = 1$. Then $\sum_{i=0}^{k} a_i 2^i \geq 2^{k+1}$. But this is impossible, because $\sum_{i=0}^{k} a_i 2^i \leq \sum_{i=0}^{k} 2^i = 2^{k+1} - 1$. A similar argument works if $a_{k+1} = 1$ and $b_{k+1} = 0$.

Now, since $a_{k+1} = b_{k+1}$ and $\sum_{i=0}^{k+1} a_i * 2^i = \sum_{i=0}^{k+1} b^i * 2^i$, it follows that $\sum_{i=0}^{k} a_i * 2^i = \sum_{i=0}^{k} b^i * 2^i$. By the induction hypothesis, $a_i = b_i$ for each i with $0 \leq i \leq k$. Therefore, $a_i = b_i$ for each i with $0 \leq i \leq k + 1$.

2.55. The basis step is obvious. Suppose the statement is true for $n = k$, and consider a sequence $a_1, a_2, \ldots, a_{k+1}$, and a permutation $b_1, b_2, \ldots, b_{k+1}$. We suppose, with no loss of generality, that a_{k+1} is the largest of the a_i's. If $b_{k+1} = a_{k+1}$, then the result we want follows easily from the induction hypothesis, since $b_1, \ldots, b_k$ is a permutation of $a_1, \ldots, a_k$. Otherwise, let $b_{k+1} = a_p$ and $a_{k+1} = b_q$.

$$(*) \quad \sum_{i=1}^{k+1} \frac{a_i}{b_i} - 1 = \left(\sum_{i=1}^{q-1} \frac{a_i}{b_i} + \frac{a_q}{a_p} + \sum_{i=q+1}^{k} \frac{a_i}{b_i}\right) + \left(\frac{a_{k+1}}{a_p} - \frac{a_q}{a_p} + \frac{a_q}{b_q} - 1\right)$$

In the right-hand expression of $(*)$, the denominators of the terms within the first pair of parentheses form a permutation of the numerators, because we have replaced the denominator b_q (i.e., a_{k+1}) by the one that appeared with a_{k+1} (i.e., a_p). The terms in the second pair of parentheses can be rewritten

$$\frac{a_{k+1} - a_q}{a_p} + \frac{a_q - a_{k+1}}{b_q} = (a_{k+1} - a_q)(\frac{1}{a_p} - \frac{1}{a_{k+1}})$$

By the induction hypothesis, the first parenthetical expression in $(*)$ is at least k, and since a_{k+1} is the largest a_i, the second is nonnegative. Therefore, the sum $\sum_{i=1}^{k+1}(a_i/b_i)$ is at least $k+1$. Furthermore, it equals $k+1$ only if the first parenthetical expression in $(*)$ is k and the second is 0, and by the induction hypothesis, this is true only if each of the a_i's is equal to the corresponding b_i.

2.56. (a) We can prove that for any $n \geq 0$, if the loop is iterated n times, then P is still true after the nth iteration. In the induction step, if the loop is iterated $k+1$ times, then B must still have been true after the kth iteration (because otherwise the loop would have terminated before iteration $k+1$). By the induction hypothesis, P is true after the kth iteration. Therefore, using the fact that P is a loop invariant, P is still true after iteration $k+1$.

(b) Let P be the condition $(r \geq 0) \wedge (x = q * y + r)$. Suppose this is true when $r = r_0$ and $q = q_0$, and that B is true (i.e., $r_0 \geq y$). Then the loop is iterated one more time. The new value of q is $q_0 + 1$, and the new value of r is $r_0 - y$. Since $r_0 \geq y$, $r = r_0 - y \geq 0$, and $x = q_0 * y + r_0 = (q_0 + 1) * y + (r_0 - y)$. Therefore, P is still true, and P is a loop invariant. Since the value of r is decreased by y in each iteration, and $y > 0$, the condition $r \geq y$ will eventually fail, so that the loop terminates. It follows from part (a) that P will be true, and the failure of the condition $r \geq y$ means that $0 \leq r < y$ will be true.

2.57. **Induction hypothesis.** $k \geq 1$, and for every n with $0 \leq n \leq k$, $f(n)$ is the largest power of 2 less than or equal to n.

Statement to be proved in induction step. $f(k+1)$ is the largest power of 2 less than or equal to $k+1$.

Proof. If $k+1$ is either $2j$ or $2j+1$, then $f(k+1) = 2f(j)$. By the induction hypothesis, $f(j)$ is the largest power of 2 less than or equal to j, say 2^p, which implies that $f(j) = 2^p \leq j < 2^{p+1}$. Therefore, $f(k+1) = 2 * 2^p = 2^{p+1}$. Since $j < 2^{p+1}$, both $2j$ and $2j+1$ are less than 2^{p+2}, so that no matter whether $k+1$ is even or odd, $f(k+1)$ is the largest power of 2 less than or equal to $k+1$.

2.58. **Proof of induction step** (using ordinary induction on k).

$$\begin{aligned} T(2^{k+1}) &\leq C(2^{k+1}) + 2T(2^{k+1}/2) \\ &\leq C(2^{k+1}) + 2(2^k * (Ck+1)) \end{aligned}$$

$$\begin{aligned} &= 2^{k+1}(C + Ck + 1) \\ &= 2^{k+1}(C(k+1) + 1) \end{aligned}$$

2.59. (a) We use structural induction on the string y.
Basis step. For any string x, $(x\Lambda)^r = x^r = \Lambda x^r = \Lambda^r x^r$. Thus the statement holds when $y = \Lambda$.
Induction hypothesis. y is a string, and for any string x, $(xy)^r = y^r x^r$.
Statement to be shown in induction step. For any string x and any $a \in \Sigma$, $(x(ya))^r = (ya)^r x^r$.
Proof. $(x(ya))^r = ((xy)a)^r = a(xy)^r = a(y^r x^r) = (ay^r)x^r = (ya)^r x^r$. The first equality is because concatenation is associative, the second is the definition of the reverse function, the third is the induction hypothesis, the fourth is also the associativity of concatenation, and the fifth is the definition of the reverse function.

(b) **Proof of induction step** (using structural induction). $((xa)^r)^r = (ax^r)^r = (x^r)^r a^r = xa^r = x(\Lambda a)^r = x(a\Lambda^r) = x(a\Lambda) = xa$. The first equality uses the definition of the reverse function. The second uses the result in part (a).

(c) **Proof of induction step** (using ordinary induction on n). $(x^{k+1})^r = (x^k x)^r = x^r(x^k)^r = x^r(x^r)^k = (x^r)^{k+1}$. The second inequality uses the result in part (a), the third the induction hypothesis.

2.60. First, to show that if $x \in pal$, then $x^r = x$. This is by structural induction. Both Λ and a have this property, for any $a \in \Sigma$.

If x is an element of *pal* for which $x^r = x$, and $a \in \Sigma$, then $(axa)^r = a(ax)^r = a(x^r a^r) = axa$. The first equality uses the definition of reverse, the second uses the result in part (a) of Exercise 2.59.

Now we must show that if $x^r = x$, then $x \in pal$. In this proof, there appears to be no real advantage in using structural induction, and we use strong induction on $|x|$.
Basis step. $\Lambda \in pal$. This is true by definition of *pal*.
Induction hypothesis. $k \geq 0$, and for any x with $|x| \leq k$ satisfying $x^r = x$, $x \in pal$.
Statement to be shown in induction step. For any x with $|x| = k+1$ satisfying $x^r = x$, $x \in pal$.
Proof. If $k+1 = 1$, $x \in pal$ because of the definition. Assume, therefore, that $k+1 \geq 2$. Then $x = ayb$ for some $a, b \in \Sigma$ and some y with $|y| = k-1$. Because $x^r = x$, $x^r = (ayb)^r = ayb$. But $(ayb)^r = by^r a$, and so we may conclude that $a = b$ and $y^r = y$. It follows then from the induction hypothesis that $y \in pal$, and so $x = aya \in pal$ because of the definition of *pal*.

2.61. The proof that no element of L contains the substring aab is a straightforward structural induction proof based on the recursive definition of L. We show the converse, that every string not containing aab is an element of L, using strong induction on $|x|$. In the basis step, $\Lambda \in L$, by definition of L.
Induction hypothesis. $k \geq 0$, and any string x that satisfies $|x| \leq k$ and does not contain the substring aab is in L.

Statement to be shown in induction step. If $|x| = k + 1$ and x does not contain aab, then $x \in L$.
Proof. In the case where $x = ya$ for some y, since x does not contain aab, y cannot. By the induction hypothesis, $y \in L$. Therefore, according to the definition of L, $x = ya \in L$. Similarly in the cases where $x = by$ and $x = aby$. To complete the proof, therefore, it is sufficient to show that one of these cases must hold. If not, then $x = aayb$ for some y. In this case, however, the first b in the string x must be preceded by aa, which is impossible.

2.62. (a) The set of strings not containing the substring aaa.
(b) The set of strings not containing the substring baa.

2.63. (a) The proof is by structural induction, using the recursive definition of L_1. The string Λ is in L_2. If x is any string in L_1 that is known to be in L_2, and $y \in L$, then $y \in L_2$ by the second statement in the definition of L_2, and so $xy \in L_2$ by the third statement in the definition of L_2.

(b) We show the following statement $P(y)$ for every $y \in L_1$: for every $x \in L_1$, $xy \in L_1$. The proof is by structural induction using the definition of L_1. $P(\Lambda)$ is clearly true. If $P(y)$ is true for some $y \in L_1$, then we must show that $P(yz)$ is true for every $z \in L$. $P(yz)$ is the statement: for every $x \in L_1$, $x(yz) \in L_1$. However, $x(yz) = (xy)z$; $xy \in L_1$ by the induction hypothesis, and so $(xy)z \in L_1$ according to the definition of L_1.

(c) The proof is by structural induction using the definition of L_2.
Basis step. To show that $\Lambda \in L_1$ and any element of L is in L_1. The first statement is true by definition of L_1, and the second follows from the second statement in the definition of L_1, using $x = \Lambda$.
Induction hypothesis. x and y are elements of L_2 that are in L_1.
Induction step. To show that $xy \in L_1$. Since x and y are in L_1 by the induction hypothesis, this follows from part (b).

2.64. **Basis step.** If $|x| = 1$ and x has more a's than b's, then $x \in L$. This is true because x must be the string a, which is in L by definition.
Induction hypothesis. $k \geq 1$, and if x is any string with more a's than b's for which $|x| \leq k$, then $x \in L$.
Statement to be shown in induction step. If $|x| = k + 1$ and x has more a's than b's, then $x \in L$.
Proof. If x contains no b's, then $x = ay$, where $|y| = k$ and y also contains more a's than b's. By the induction hypothesis, $y \in L$, and so by part 2 of the definition, $x = ay \in L$. The three remaining cases we consider are those in which x starts with b, x ends with b, and x starts and ends with a but contains at least one b.

Let $d(z) = n_a(z) - n_b(z)$, the difference between the number of a's and the number of b's in the string z. If x starts with b, then $x = by$ for some y satisfying $d(y) \geq 2$. Consider prefixes of y: the shortest, Λ, satisfies $d(\Lambda) = 0$; the longest, y, satisfies $d(y) \geq 2$; since adding 1 to the length of the prefix changes the value of d by 1, there must be some prefix y_1 for which $d(y_1) = 1$. Therefore, if $y = y_1y_2$, we have $d(y_2) \geq 1$. By the induction hypothesis, both y_1 and y_2 are in L. Therefore, the string $x = by_1y_2$ is also in L.

The proof in the case when x ends with b is similar. Suppose $x = aya$, where y contains m b's and $m \geq 1$. If x_1 is the portion of x preceding the last b, and $d(x_1) > 0$, then $x = x_1 b x_2$ where $d(x_1)$ and $d(x_2)$ are both positive; the induction hypothesis implies that x_1 and x_2 are in L, and it follows from the definition of L that x is. Otherwise, $d(x_1) \leq 0$. In this case, for each i with $1 \leq i \leq m$, let w_i be the prefix of x preceding the ith b and z_i the suffix following the ith b. Then $d(w_m) \leq 0$; let j be the *smallest* i for which $d(w_i) \leq 0$. Since x starts with a, j must be at least 2. We know, therefore, that $d(w_{j-1}) > 0$ and $d(w_j) \leq 0$. However, w_j has only one more b than w_{j-1}, which implies that $d(w_{j-1}) = 1$. Now we have $x = w_{j-1} b z_{j-1}$, where $d(x) > 0$ and $d(w_{j-1}) = 1$. It follows that $d(z_{j-1})$ is also positive. Now we can apply the induction hypothesis to w_{j-1} and z_{j-1}; they are both in L, and so the string x is also, by the definition of L.

2.65. The proof that every element of L has equal numbers of a's and b's is a straightforward structural induction proof. The converse is proved by strong induction on $|x|$.
Basis step. $\Lambda \in L$. This is true by definition of L.
Induction hypothesis. $k \geq 0$, and any string x with $|x| \leq k$ having equal numbers of a's and b's is in L.
Statement to be shown in induction step. If x has equal numbers of a's and b's, and $|x| = k + 1$, then $x \in L$.
Proof. If x is either $01y$ or $10y$, for some y with $|y| = k - 1$, then y has equal numbers of a's and b's, and is in L by the induction hypothesis. Therefore, x, either $0\Lambda1y$ or $1\Lambda0y$, is in L by definition of L.

Otherwise x starts with either 00 or 11. We complete the proof in the first case, and the second case is similar. Let $d(z) = n_0(z) - n_1(z)$, and consider $d(z)$ for the prefixes z of x. $d(00) = 2$ and $d(x) = 0$. Since adding a single symbol to a prefix changes the d value by 1, there must exist a prefix z longer than 00 for which $d(z) = 1$. Let z_1 be the *longest* such prefix. Then x must be $z_1 1 z_2$ for some string z_2—otherwise, since $d(z_1 0) = 2$, there would be a longer prefix for which $d = 1$. We now know that $x = 00y_1 1 z_2$, and $d(0y_1) = d(00y_1 1) = 0$. Therefore, $d(z_2) = 0$ as well. By the induction hypothesis, both $0y_1$ and z_2 are in L. Therefore, $x = 0(0y_1)1z_2$ can be also, because of the definition of L.

2.66. We show that for $n \geq 0$ and elements $x_1, \ldots, x_n$ of S, if $e^*(x_1 \ldots x_n) = e^*(y)$ for some $y \in S^*$, then $y = x_1 x_2 \ldots x_n$. (Note: although the problem doesn't say explicitly that $e^*(\Lambda) = \Lambda$, this is the correct interpretation of what $x_1 x_2 \ldots x_n$ and $e(x_1) \ldots e(x_n)$ mean when $n = 0$.)
Basis step. We must show that if $e^*(\Lambda) = \Lambda = e^*(y)$ for some $y \in S^*$, then $y = \Lambda$. This is true because $\Lambda \notin T$.
Induction hypothesis. $k \geq 0$, and for any $x_1, \ldots, x_k$ in S, if $e^*(x_1 \ldots x_k) = e^*(y)$ for some $y \in S^*$, then $x_1 \ldots x_k = y$.
Statement to be shown in induction step. For any $x_1, \ldots, x_{k+1}$ in S, if $e^*(x_1 x_2 \ldots x_{k+1}) = e^*(y)$ for some $y \in S^*$, then $x_1 x_2 \ldots x_{k+1} = y$.
Proof. Suppose $e(x_1)e(x_2) \ldots e(x_{k+1}) = e(y_1)e(y_2) \ldots e(y_m)$. Then $m \geq 1$. Now if the two strings $e(x_1)$ and $e(y_1)$ are not equal, one must be a prefix of the other, and this is

impossible; therefore, $e(x_1) = e(y_1)$. Now we have $e(x_2)\dots e(x_{k+1}) = e(y_2)\dots e(y_m)$. The induction hypothesis now implies that $x_2 \dots x_{k+1} = y_2 \dots y_m$, and this implies the result.

2.67. Nothing happens until the nth day; on that day all n wives of unfaithful husbands realize that their husbands are unfaithful, and proceed to kill them. To show this, we may let $P(n)$ be the statement that if there are at least n unfaithful husbands, no one has been killed by the end of the $n-1$th day, and if there are exactly n, then all n are killed on the nth day. Suppose that this is true when $n = k$, and now suppose $n = k+1$. Consider the wife of one of one of the unfaithful husbands. She knows that there are either k unfaithful husbands (if her husband is faithful) or $k+1$ (if he is not). Furthermore, she realizes that if her husband is faithful, the k wives of unfaithful husbands will all kill their husbands on the kth day, and if he is not, they won't. At midnight of the kth day, however, no one has been killed, and she concludes that her husband must be unfaithful.

2.68. Yes. Let $S(n)$ be the statement: for every $m \geq 0$, $P(m,n)$ is true. We can prove that $S(n)$ is true for every $n \geq 0$, by induction.
Basis step. We must show that for every $m \geq 0$, $P(m,0)$ is true. This is itself an induction proof. The basis step uses the fact that $P(0,0)$ is true, and the induction step uses the fact that if $P(k,0)$ is true, then $P(k+1,0)$ is true.
Induction hypothesis. $k \geq 0$, and for every $m \geq 0$, $P(m,k)$ is true.
Statement to be shown in induction step. For every $m \geq 0$, $P(m,k+1)$ is true.
Proof. This follows immediately from the assumption on P: if $P(m,k)$ is true, $P(m,k+1)$ is true.

2.69. Let us say that x satisfies property p if $x = 2$ or x is divisible by 5. Then $\{x \in S \mid x$ satisfies property $P\}$ is the set of all multiples of 5. However, $\{x \in \mathcal{N} \mid x$ has property $P\}$ is the set containing 2 and all multiples of 5. This second set is not closed under the operation of adding 5. Therefore, the proof using structural induction that every element of S satisfies property P could not be simplified as in the two examples.

2.70. Assume for the sake of this discussion that $\circ$ is a binary operation. Consider the algorithm described by this pseudocode.

```
A = I;
Repeat
  B = A;
  for each element x of A
    for each element y of A
      B = B ∪ {x ∘ y}
until B = A;
Determine whether the desired element is in A
```

The algorithm terminates because U is finite: there can be only a finite number of iterations of the Repeat loop in which one or more elements are added to A. The final value of A is S, because if T is any subset of U that contains all the elements of I and is closed under

o, any element added to A in any of the iterations must belong to T.